Problem Seminar. Fall 2015.

Problem Set 3. The Pigeonhole principle.

The Pigeonhole principle.

If $n+1$ objects are distributed among n boxes, then some box will contain at least 2 objects.

Classical results.

1. Given n integers, prove that some non-empty subset has sum divisible by n.
2. Let A be a set of $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$. Show that some element of A divides another.
3. Prove that every sequence of n^2 distinct numbers contains a monotone (monotonically increasing or decreasing) subsequence of length n.

Problems.

1. **Put 2002. A2.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
2. **Put 1995. B1.** For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]
3. **Put 2006. A3.** Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.
4. **Put 1994. A4.** Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.
5. **US 2000.** Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.
6. **Put 1980. B4.** Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)