Problem Seminar. Fall 2015.

Problem Set 2. Induction.

Classical results.

1. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

2. **Fermat's little theorem.** Let p be a prime number, and n a positive integer. Show that $n^p - n$ is divisible by p.

3. **Ramsey's theorem.** Show that for any pair of positive integers (r, s), there exists a positive integer $R(r, s)$ such that if the edges of a complete graph on $R(r, s)$ vertices are coloured red or blue, then either there exists a complete subgraph on r vertices which is entirely blue, or a complete subgraph on s vertices which is entirely red. (A *complete graph* is a graph where every two vertices are connected by an edge.)

4. Prove that the Fibonacci sequence satisfies the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

for $n \geq 0$ (The Fibonacci sequence F_n is defined by $F_0 = 1, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.)

Problems.

1. **Put 2005. A1.** Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

2. **USA 1997.** An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

3. **GA 32.** Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq (1 + \sqrt[n]{a_1 a_2 \ldots a_n})^n.$$

4. **IMO 2001. B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

5. **Putnam 2006. B3.** Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

6. **Putnam 1996. A4.** Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;

   (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

7. **Putnam 1996. A6.** A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.