Problem Seminar. Fall 2015.

Problem Set 1. Proofs by contradiction.

Classical results.

1. Prove that there are infinitely many prime numbers. (Recall that an integer $p > 1$ is *prime* if its only divisors are p and 1.)

2. Prove that there exists no polynomial P with integer coefficients such that $P(n)$ is prime for every positive integer p.

3. Recall that
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots.$$
Show that e is irrational.

Problems.

1. **Putnam 1952. A1.** The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.

2. **Putnam 1995. A1.** Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

3. **Germany 1985.** Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

4. **Put 2006. A2.** Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10, then Alice can take the remaining stones to win.)

5. **IMC 2014. A4.** Let $n > 6$ be a perfect number, and let $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ be its prime factorisation with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number. (A number n is perfect if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

6. **Putnam 1996. B4.** For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

7. **Putnam 1958. B5.** S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

8. **Putnam 2002. A6.** Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?