Problem Solving seminar - Team selection contest

1. A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n^{th} time (so $f(0) = 0$). Find a formula for $f(n)$.

2. Two circles α, β touch externally at the point X. Let A, P be two distinct points on α different from X, and let AX and PX meet β again in the points B and Q respectively. Prove that AP is parallel to QB.

3. Find the maximum value of $xy^3 + yz^3 + zx^3 - xz^3 - yx^3 - zy^3$, where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

4. Let $n \geq 2$ be a positive integer and let $f(x)$ be the polynomial

$$1 - (x + x^2 + \ldots + x^n) + (x + x^2 + \ldots + x^n)^2 - \ldots + (-1)^n (x + x^2 + \ldots + x^n)^n.$$

If r is an integer such that $2 \leq r \leq n$, show that the coefficient of x^r in $f(x)$ is zero.

5. Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that

$$\sum_{n=1}^{\infty} \left| 1 - \frac{a_{n+1}}{a_n} \right|$$

is divergent.

6. One thousand students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ and a contiguous group of $2k$ students in the circle, such that the first half of the group contains the same number of girls as the second half.