MATH 350: Graph Theory and Combinatorics. Fall 2015.
Due in class on Friday, October 2nd.

Assignment #1: Paths, Cycles and Trees.

1. For each of the following statements decide if it is true or false, and either prove it or give a counterexample.

a) If u, v, w are vertices of G, and there is an even length path from u to v and an even length path from v to w then there is an even length path from u to w.

b) If G is connected and has no path with length larger than k, then every two paths in G of length k have at least one vertex in common.

c) If u, v, w are vertices of G, and there is a cycle of G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

d) If e, f, g are edges of G, and there is a cycle containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

2. Let $d_1, d_2, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

3. Let G be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there exists a path in G from u to v of length at most k. Show that either G contains a cycle of length $\leq 2k + 1$ or G is a tree.

4. Let T be a tree with l leaves. Let k be a positive integer with $2k \geq l$. Show that there exists paths $P_1, P_2, \ldots, P_k$ such that

(i) $P_1 \cup P_2 \cup \ldots \cup P_k = T$,

(ii) $V(P_i) \cap V(P_j) \neq \emptyset$ for all i, j.

5. Let T be a tree, and let $T_1, \ldots, T_n$ be connected subgraphs of T so that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$. Show that $V(T_1 \cap T_2 \cap \ldots \cap T_n) \neq \emptyset$. [*Hint*: Delete a leaf and use induction on $|V(T)|$.]