MATH 350: Graph Theory and Combinatorics. Fall 2015.

Assignment #5: Series-parallel and Planar graphs. Solutions.

1. A graph G is *outerplanar* if it can be drawn in the plane so that every vertex is incident with the infinite region. Show that a graph G is outerplanar if and only if G has no K_4 or $K_{2,3}$ minor.

Solution: Let G' be obtained from G by adding an extra vertex to G adjacent to every other vertex. Then G' is planar if and only if G is outerplanar and it is easy to cherk that G contains a K_4 or $K_{2,3}$ minor if and only if G' contains a K_5 or $K_{3,3}$ minor. Thus the problem follows from Kuratowski's theorem (18.3).

2.

a) Show that every series-parallel graph is planar.

b) Is every series-parallel graph outerplanar?

c) What is the maximum possible number of edges in a simple series-parallel graph with n vertices?

Solution:

a) A graph is series parallel if and only if it does not contain K_4 as a minor, and a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. As K_4 is a minor of both K_5 and $K_{3,3}$ the result follows.

b) No, by the first problem in this assignment $K_{2,3}$ is not outerplanar, but it is easy to check that it is series-parallel.

c) The answer is $2n-3$ for $n \geq 3$. This bound can be achieved by taking a graph with 2 vertices joined by $n-1$ parallel edges and subdividing $n-2$ of those edges once.

We show that $|E(G)| \leq 2|V(G)| - 3$ for every series-parallel graph G with $|V(G)| \geq 3$ by induction on $|V(G)|$. The base case is clear. For the induction step, by Lemma 16.2, G has a vertex v of degree two. Thus we have

$$|E(G)| \leq |E(G \setminus v)| + 2 \leq 2|V(G \setminus v)| - 3 + 2 = 2|V(G)| - 3,$$

using the induction hypothesis for $G \setminus v$.

3. Let G be a loopless graph, such that G does not contain $K_{2,3}$ as a minor. Show that either $\chi(G) \leq 3$, or G contains K_4 as a subgraph.

Solution: If G does not contain K_4 as a minor, then $\chi(G) \leq 3$ by Corollary 16.3. If G contains K_4 as a minor, but does not contain K_4 as a subgraph, then G contains a graph K_4^+, obtained from K_4 by subdividing one of its edges, as a minor. (As shown in the proof of Theorem 18.4, if G contains a graph H with maximum degree three as a minor then G contains a subdivision of G as a subgraph.) The graph K_4^+ contains $K_{2,3}$ as a subgraph, and therefore G contains $K_{2,3}$ as a minor, yielding the desired contradiction.

4. Let G be a graph drawn in the plane. Suppose that there exists a vertex v so that v belongs to the boundary of every region. Show that

$$\alpha(G) \geq \frac{1}{2}(|V(G)| - 1).$$

Solution: We have $\text{Reg}(G \setminus v) = 1$ by the assumption. Thus $G \setminus v$ is a forest and 2-colorable. It follows that

$$\alpha(G) \geq \alpha(G \setminus v) \geq \frac{|V(G \setminus v)|}{2} = \frac{1}{2}(|V(G)| - 1).$$

5. Let G be drawn in the plane so that

- the boundary of the infinite region is some cycle C,
- every other region has boundary a cycle of length 3, and
- every vertex of G not in C has even degree.

Show that $\chi(G) \leq 3$.

Solution: Following the hint, we prove the result by induction on $|V(G)|$. The induction base $|V(G)| = 3$ is trivial. For the induction step, suppose first that some two vertices of C are joined by an edge $e \notin E(C)$. We can express $C + e$ as a union of two cycles C_1 and C_2. Let G_1 and G_2 be the subgraphs of G bounded by C_1 and C_2, respectively. Then G_1 and G_2 are 3-colorable by the induction hypothesis and we can combine their colorings to produce a coloring of G, as $G_1 \cap G_2$ consists of a pair of adjacent vertices.

Suppose now that G contains no edge e as above. Consider $v \notin V(C)$. Assume first that there are no parallel edges incident to v. Let $u_0, u_1, \ldots, u_k$ be the neighbors of v, listed in the order that the edges incident to v appear around it, with $u_0, u_k \in V(C)$. In a graph $G \backslash v$ the infinite face is bounded by a cycle obtained from C by replacing v with $u_0, u_1, \ldots, u_k$. By the induction hypothesis there exists a 3-coloring $c : V(G \backslash v) \to \{1, 2, 3\}$. Note that for each vertex $u \in V(G \backslash v)$ the colors of its neighbors alternate as we enumerate these neighbors in the order edges incident to u appear around u. As $\deg(u_i)$ is even for $i = 1, 2, \ldots, k-1$ we deduce that $c(u_{i-1}) = c(u_{i+1})$ for each such i. It follows that only two colors are used on $u_0, u_1, \ldots, u_k$ and thus c can be extended to v.

Suppose, finally, that some vertex $v \in V(C)$ is joined to another vertex $u \in V(G)$ by a pair of parallel edges e and f. (The graph G does not contain any loops as every finite face is bounded by a cycle of length 3.) Let R be the region of the plane bounded by e and f. Let G_1 be the subgraph of G drawn within R. Let G_2 be obtained from G by deleting the vertices within the interior of R and the edge f. Applying the induction hypothesis to G_1 and G_2 we obtain a valid coloring of G, as in the previous paragraph. The only caveat is that we need to verify that the degree of u is even in G_2.

Suppose not. Then the degree of u is even in G_1. Consider the graph G_1^* dual to G_1. In G_1^* every region is bounded by an even cycle. Thus G_1^* is bipartite, but every vertex of G_1^* has degree 3, except for one vertex of degree 2, corresponding to the infinite face of G_1. Summing degrees of the vertices on either side of the bipartition we obtain is contradiction, as one of the sums will be divisible by 3, but not the other. This contradiction finishes the proof.