MATH 350: Graph Theory and Combinatorics. Fall 2015.

Assignment #4: Ramsey theorem, matching and vertex coloring. Solutions.

1. Show that $R(3,4) = 9$.

Solution: First we show that $R(3,4) \leq 9$. Let G be a graph with $|V(G)| = 9$. Our goal is to show that $\alpha(G) \geq 3$ or $\omega(G) \geq 4$. As $R(3,3) = 6$, if some vertex v of G has at least 6 neighbors, the neighborhood of v contains an independent set of size 3 or a clique of size 3, implying the required statement for G. Similarly, as $R(2,4) = 4$, we are if some vertex of G has ≥ 4 non-neighbors. Thus we may assume that every vertex of G has exactly 5 neighbors and exactly 3 non-neighbors. But this last case is impossible as every graph contains odd number of vertices of even degree.

To show that $R(3,4) > 8$, consider the graph G obtained from the cycle with vertices $v_1, v_2, \ldots, v_8$ in cyclic order by adding edges v_1v_5 and v_2v_6. As the length eight cycle contains exactly two independent sets of size four, namely $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6, v_8\}$, we have $\alpha(G) = 3$. It is also easy to see that $\omega(G) = 2$, that is G has no K_3 subgraph. It follows that $R(3,4) = R(4,3) > 8$.

2. Let

$$r_k := R_k(\underbrace{3, 3, \ldots, 3}_{k \text{ times}}).$$

(I.e. r_k is the minimum integer $n > 0$ such that every coloring of edges of K_n in k colors is guaranteed to produce a monochromatic triangle.) Show that

$$r_k \leq k(r_{k-1} - 1) + 2$$

for $k \geq 2$.

Solution: We need to show that for $n = k(r_{k-1} - 1) + 2$ every coloring of edges of K_n in k colors is guaranteed to produce a monochromatic triangle. Let v be an arbitrary vertex of K_n at least $\lceil \frac{n-1}{k} \rceil = r_{k-1}$ edges incident with v have the same color, say k. Let X be the set of vertices joined to v by edges of this color. If two vertices of X are joined by an edge of color k they form a monochromatic triangle with v. Otherwise, only $k-1$ colors are used on edges joining the vertices of X, and the subgraph induced on X contains a monochromatic triangle by definition of r_{k-1}.

3. Let G be a graph and $Z \subseteq V(G)$. Show that the following are equivalent:

(i) G has a matching covering Z, and

(ii) for every $X \subseteq V(G)$ there are at most $|X|$ odd components C of $G \backslash X$ such that $V(C) \subseteq Z$.

Solution: (i) $\Rightarrow$ (ii): Let M be a matching covering Z. For every $X \subseteq V(G)$ and every component C of $G \backslash X$, some edge of M joins a vertex of X to a vertex of C. Thus there at least as many vertices as there are such odd components.

(ii) $\Rightarrow$ (i): Let G' be obtained from G by adding a set Y of $|V(G)|$ extra vertices, so that the vertices of Y are pairwise adjacent and every vertex of Y is adjacent to every vertex in $V(G) - Z$. Any matching in G covering Z can be extended to a perfect matching in G'. Conversely, every perfect matching of G' contains a matching in G covering Z. Thus by Tutte's theorem (13.1) (i) holds as long as the following condition holds:

(iii) $G' \backslash X'$ has at most $|X'|$ odd components for every $X' \subseteq V(G')$.

Condition (iii) is trivially satisfied when $Y \subseteq X'$ and we assume that $Y \not\subseteq X'$. All the vertices of $V(G') - Z - X'$ belong to the same component of $G' \backslash X'$. Let $X = X' \cap V(G)$. By (ii) and the preceding observation $G' \backslash X'$ has at most $|X| + 1$ odd components. As in the proof of Theorem 13.1 the number of odd components of $G' \backslash X'$ has the same parity as $|X'|$. Therefore (iii) holds, as desired.

4. Show that if G is a loopless graph, $k \geq 1$ is an integer and $\chi(G) > k$ then G has a path with k edges.

Solution: Order the vertices of G arbitrarily: $v_1, v_2, \ldots, v_n$. We apply the coloring algorithm with respect to this ordering. If v_i receives color c then it has a neighbor v_j with $j < i$ such that v_j receives color $c - 1$. This observation easily implies by induction on c that every vertex v_i which receives color c is an end of some path P with $c - 1$ edges such that $V(P) \subseteq \{v_1, v_2, \ldots, v_i\}$. Thus, if a vertex receives a color $k + 1$ then it is an end of a path with k edges, as desired.

5. Let G be a loopless graph with $\chi(G) = k$ for some positive integer k. Show that G contains at least k vertices with degree at least $k - 1$.

Solution: Order the vertices of G in the non-increasing order of degrees and apply the coloring algorithm to this ordering. If a vertex v receives color k then it must be preceded by $\geq k - 1$ of its neighbors in the ordering. Thus v has degree $\geq k - 1$ and so does each of its $\geq k - 1$ neighbors preceding it in the ordering.