MATH 350: Graph Theory and Combinatorics. Fall 2015.

Assignment #3: Menger's theorem, vertex covers and network flows.

1. Show that $\tau(G) \leq \frac{1}{2}(|E(G)|+1)$ for every connected graph G.

Solution: By induction on $|V(G)|$. Base case for $|V(G)| \leq 2$ is routine. Induction step (for $|V(G)| \geq 3$): Let T be a spanning tree of G. Let L be the set of leaves of T. If for some $u \in L$, the degree of u in G is at least two, we apply the induction hypothesis to $G \setminus v$ to obtain

$$\tau(G) \leq \tau(G \setminus v) + 1 \leq \frac{1}{2}\left(|E(G \setminus v)|+1\right) + 1 \leq \frac{1}{2}\left(|E(G)|+1\right),$$

as desired. Otherwise, consider a leaf w of $T \setminus L$. (We have $V(T \setminus L) \neq \emptyset$, as $|V(G)| \geq 3$.) Let L' be the set of leaves in L adjacent to w. Let $G' := G \setminus w \setminus L'$. Note that, G' is connected and $|E(G')| \leq |E(G)| - 2$. Further, if X is vertex cover of G' then $X \cup \{w\}$ is a vertex cover of G,. We apply the induction hypothesis to G' to obtain,

$$\tau(G) \leq \tau(G') + 1 \leq \frac{1}{2}\left(|E(G')|+1\right) + 1 \leq \frac{1}{2}\left(|E(G)|+1\right),$$

completing the proof.

2. Let v be a vertex in a 2-connected graph G. Show that v has a neighbor u such that $G \setminus u \setminus v$ is connected.

Solution: Let U be the set of neighbors of v in G. Let T be the minimum connected subgraph of $G \setminus v$ such that $U \subseteq V(T)$. It is easy to see that T is a tree and that every leaf of T is a neighbor of v. Let u be a leaf of T. Then $T \setminus u$ is connected. Suppose for a contradiction that $G \setminus u \setminus v$ is not connected and consider a component C of $G \setminus u \setminus v$ which does not contain $T \setminus u$. Thus C contains no neighbor of v and so it is a connected component of $G \setminus u$. It follows that $G \setminus u$ is not connected, contradicting 2-connectivity of G.

3. Let G be a connected graph in which every vertex has degree three. Show that if G has no cut-edge then every two edges of G lie on a common cycle.

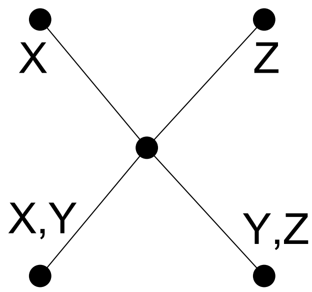

Figure 1: Counterexample for Problem 6b).

Solution: Note that G is loopless, as otherwise it would contain a cut-edge. Consider $e_1, e_2 \in E(G)$ and let x_i, y_i be the ends of e_i for $i = 1, 2$. If there exist two vertex-disjoint paths from x_1, y_1 to x_2, y_2 then these paths together with e_1 and e_2 form the required cycle. Otherwise, by Menger's theorem, there exists a separation (A, B) of order 1 with $x_1, y_1 \in A$, $x_2, y_2 \in B$. Let $\{v\} = A \cap B$. Let u_1, u_2, u_3 be the other ends of edges incident to v. (These three vertices are not necessarily distinct.) Without loss of generality, $u_1 \in A, u_2, u_3 \in B$. Then the edge joining u_1 and v is a cut-edge, a contradiction.

4.

a) Distinct $u, v \in V(G)$ are *k-linked* if there are k paths $P_1, ..., P_k$ of G from u to v so that $E(P_i \cap P_j) = \emptyset$ $(1 \le i < j \le k)$. Suppose u, v, w are distinct and u, v are k-linked, and so are v, w. Does it follow that u, w are k-linked?

Solution: Yes. By Theorem 10.4, if u and w are not k-linked then there exists $X \subseteq V(G)$ with u in X, $w \notin X$ and $|\delta(X)| < k$. By symmetry, we may assume $v \in X$. Then the opposite direction of Theorem 10.4 implies that v and w are not k-linked.

b) Subsets $X, Y \subseteq V(G)$ are *k-joined* if $|X| = |Y| = k$ and there are k paths $P_1, ..., P_k$ of G from X to Y so that $V(P_i \cap P_j) = \emptyset$ $(1 \le i < j \le k)$. Suppose $X, Y, Z \subseteq V(G)$ and X, Y are k-joined, and so are Y, Z. Does it follow that X, Z are k-joined?

Solution: No. See Figure 1 for an example with $k = 2$.

5. Let G be a directed graph and for each edge e let $\phi(e) \ge 0$ be an integer, so that for every vertex v,

$$\sum_{e\in\delta^-(v)} \phi(e) = \sum_{e\in\delta^+(v)} \phi(e)$$

Show there is a list $C_1, ..., C_n$ of directed cycles (possibly with repetition) so that for every edge e of G,

$$|\{i : 1 \le i \le n,\ e \in E(C_i)\}| = \phi(e).$$

Solution: Induction on $S := \sum_{e\in E(G)} \phi(e)$. Base case: $S = 0$ is trivial. For the induction step, it suffices to find a directed cycle C in G so that $\phi(e) \ge 1$ for every edge $e \in E(G)$, as one can then apply the induction hypothesis to

$$\phi'(e) := \begin{cases} \phi(e), & \text{if } e \notin E(G) \\ \phi(e) - 1, & \text{if } e \in E(G) \end{cases}$$

Let e be an edge of G with $\phi(e) \ge 1$, a tail u and a head v. Then ϕ restricted to $G \setminus e$ is a v-u-flow of value 1 and by Lemma 11.3 there exists a directed path P in $G \setminus e$ so that ϕ is positive on every edge of the path. The path P together with e forms the desired cycle.