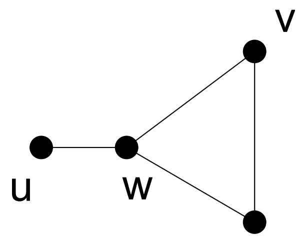

Figure 1: Counterexample for Problem 1a).

MATH 350: Graph Theory and Combinatorics. Fall 2014.

Assignment #1: Paths, Cycles and Trees. Solutions.

1. For each of the following statements decide if it is true or false, and either prove it or give a counterexample.

a) If u, v, w are vertices of G, and there is an even length path from u to v and an even length path from v to w then there is an even length path from u to w.

Solution: False. See Figure 1.

b) If G is connected and has no path with length larger than k, then every two paths in G of length k have at least one vertex in common.

Solution: True. Suppose for a contradiction that P_1 and P_2 are two vertex disjoint paths of length k. Let vertices of P_i be $v^i_1, v^i_2, \ldots, v^i_{k+1}$, in order. Let Q be the a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be a short as possible. Let v^1_n and v^2_m be the ends of Q. We can suppose without loss of generality that $m, n \geq k/2 + 1$. Then a path obtained by taking the union of the subpath of P_1 from v^1_1 to v^1_n, the path Q and the subpath of P_2 from v^2_1 to v^2_m has at least $m + n \geq k + 2$ vertices, a contradiction.

c) If u, v, w are vertices of G, and there is a cycle of G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

Solution: False. Consider a graph G with $V(G) = \{u, v, w\}$ and $E(G)$ consisting of a pair of edges joining u to v and a pair of edges joining v to w.

d) If e, f, g are edges of G, and there is a cycle containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

Solution: True. Without loss of generality we may assume that G is connected. The result follows immediately from the next claim.

Claim: If there exist does not exist a cycle containing edges e and g then there does not exist a vertex $u \in V(G)$ such that every path in G sharing one end with e and another with g contains u.

Proof: The claim trivially holds if e or g is a loop, so we assume that neither is. Let P with vertex set $v_1, v_2, \dots, v_k$, in order, be a path with e joining v_1 to v_2 and g joining v_{k-1} and v_k. Let $f_i \in E(P_i)$ be the edge with ends v_i and v_{i+1}. Let j be chosen minimum so that no cycle in G contains e and f_j. We will show that $u = v_j$ satisfies the claim.

Suppose not. Let C be a cycle containing e and f_{j-1} and let P' be a path from an end of e to an end of f avoiding u. Choose a subpath Q of P' with one end in $V(C)$ and another in $\{v_{j+1}, v_{j+2}, \dots, v_k\}$ as short as possible. Then $C \cup Q \cup P$ contains a cycle containing both e and f_j, a contradiction. (The last statement requires some case checking.)

2. Let $d_1, d_2, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \dots, d_n$ if and only if

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

Solution: "Only if" direction: By Theorems 1.1 and 3.1, if T is a tree with degrees $d_1, d_2, \dots, d_n$ then

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2|V(T)| - 2 = 2n - 2.$$

"If" direction: By induction on n. The base case $n = 2$ is trivial, as K_2 is the unique tree on two vertices. For the induction step, if $n > 2$, then $n < \sum_{i=1}^{n} d_n = 2n - 2 < 2n$, therefore at least one of the d_i's is equal to 1, and at least one of the d_i's is bigger than 1. Without loss of generality, $d_n = 1$, $d_{n-1} > 1$. By the induction hypothesis there exists a tree T' with vertex degrees $d_1, d_2, \dots, d_{n-1} - 1$. Let T be obtained from T' by adding a leaf to it with the unique neighbor of the leaf being a vertex of degree $d_{n-1} - 1$. It is easy to check that T is a tree and has degrees $d_1, d_2, \dots, d_n$.

3. Let G be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there exists a path in G from u to v of length at most k. Show that either G contains a cycle of length $\leq 2k + 1$ or G is a tree.

Solution: Clearly, G is connected. If G is not a tree then it contains a cycle. Let C be the cycle in G of smallest length and let $v_1, v_2, \ldots, v_l$ be the vertices of C in order. Suppose for a contradiction that $l > 2k+1$. Let P be the shortest path from v_1 to v_{k+1} in G. Then P has length at most k and it follows that $P \subsetneq C$. Thus there exists a subpath Q of P with ends $v_i, v_j \in V(P)$ and otherwise disjoint from C. The union of Q with each of the two paths in C with ends v_i and v_j is a cycle, and so each of these cycles must have length at least l. The sum of their lengths, however, is equal to $l + 2|E(Q)| \leq l + 2|E(P)| \leq l + 2k < 2l$, a contradiction.

4. Let T be a tree with l leaves. Let k be a positive integer with $2k \geq l$. Show that there exists paths $P_1, P_2, \ldots, P_k$ such that

(i) $P_1 \cup P_2 \cup \ldots \cup P_k = T$,

(ii) $V(P_i) \cap V(P_j) \neq \emptyset$ for all i, j.

Solution: Choose $P_1, \ldots, P_k$ so that all the leafs of T belong to $V(P_1) \cup V(P_2) \cup V(P_k)$, and, subject, to the first condition $|V(P_1)| + |V(P_2)| + \ldots + |V(P_k)|$ is maximum. (A choice satisfying the first condition is possible, as $2k \geq l$.) We claim that both (i) and (ii) hold. Indeed, suppose that $V(P_i) \cap V(P_j) = \emptyset$ for some i, j. Then there exists a unique path $Q \subseteq T$ such that Q has one end in $V(P_i)$, another end in $V(P_j)$ and is otherwise disjoint from $P_i \cup P_j$. There exists path P'_i and P'_j in T such that $P'_i \cup P'_j j = P_i \cup P_j \cup Q$. Replacing P_i and P_j byP'_i and P'_j we obtain a contradiction to our initial choice of paths. Thus (i) holds.
Suppose (ii) does not hold. Then there exists

$$e \in E(T) - (E(P_1) \cup \ldots \cup E(P_k)).$$

The two components T_1 and T_2 of $T \setminus e$ each contain a leaf of T. Therefore each of T_1 and T_2 contains at least one of the paths $P_1, \ldots, P_k$. If $P_i \subseteq T_1$ and $P_j \subseteq T_2$ then $V(P_i) \cap V(P_j) = \emptyset$, contradicting property (i), which was already established. Thus (ii) also holds.

5. Let T be a tree, and let $T_1, \ldots, T_n$ be connected subgraphs of T so that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$. Show that $V(T_1 \cap T_2 \cap \ldots \cap T_n) \neq \emptyset$.
Solution: Proof by induction on $V(T)$. Base case $|V(T)| = 1$ is trivial. For the induction step, let v be a leaf of T and let u be the unique vertex of T adjacent to v. Let $T' = T \setminus v$ and let $T'_i = T' \setminus v$ for $i = 1, 2, \ldots, n$. If $V(T'_i \cap T'_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$, then we can apply the induction hypothesis to T' to complete the proof. Thus we may assume,

without loss of generality, that $V(T_1') \cap V(T_2') = \emptyset$. It follows that $V(T_1) \cap V(T_2) = \{v\}$. Thus either $u \notin V(T_1)$ or $u \notin V(T_2)$. Without loss of generality, we have $V(T_1) = \{v\}$. Therefore $v \in V(T_i)$ for every $1 \leq i \leq n$ by the assumption and $v \in V(T_1 \cap T_2 \cap \ldots \cap T_n)$, as desired.