MATH 350: Graph Theory and Combinatorics. Fall 2013.
Due in class on Monday, October 21st.

Assignment #2: Bipartite graphs, matchings and connectivity.

1. Show that every loopless graph G contains a bipartite subgraph with at least $|E(G)|/2$ edges.

2. Let G be a loopless graph in which every vertex has degree ≥ 1. Let X be the largest matching in G, and let Y be the smallest set of edges of G so that every vertex of G is incident with ≥ 1 edge in Y. Show that $|X| + |Y| = |V(G)|$.

3. Let G be a bipartite graph with bipartition (A, B) in which every vertex has degree ≥ 1. Assume that for every edge of G with ends $a \in A$ and $b \in B$ we have $\deg(a) \geq \deg(b)$. Show that there exists a matching in G covering A.

4. Given integers $n \geq m \geq k \geq 0$, determine the maximum possible number of edges in a bipartite graph G with bipartition (A, B), with $|A| = n, |B| = m$ and no matching of size k.

5. Let v be a vertex in a 2-connected graph G. Show that v has a neighbor u such that $G \setminus u \setminus v$ is connected.

6.

a) Distinct $u, v \in V(G)$ are *k-linked* if there are k paths $P_1, ..., P_k$ of G from u to v so that $E(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose u, v, w are distinct and u, v are k-linked, and so are v, w. Does it follow that u, w are k-linked?

b) Subsets $X, Y \subseteq V(G)$ are *k-joined* if $|X| = |Y| = k$ and there are k paths $P_1, ..., P_k$ of G from X to Y so that $V(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose $X, Y, Z \subseteq V(G)$ and X, Y are k-joined, and so are Y, Z. Does it follow that X, Z are k-joined?