MATH 350: Graph Theory and Combinatorics. Fall 2013.
Due in class on Thursday, October 3rd.

Assignment #1: Paths, Cycles and Trees.

**1.** For each of the following statements decide if it is true or false, and either prove it or give a counterexample.

**a)** If $u, v, w$ are vertices of $G$, and there is an even length path from $u$ to $v$ and an even length path from $v$ to $w$ then there is an even length path from $u$ to $w$.

**b)** If $G$ is connected and has no path with length larger than $k$, then every two paths in $G$ of length $k$ have at least one vertex in common.

**c)** If $u, v, w$ are vertices of $G$, and there is a cycle of $G$ containing $u$ and $v$, and a cycle containing $v$ and $w$, then there is a cycle containing $u$ and $w$.

**d)** If $e, f, g$ are edges of $G$, and there is a cycle containing $e$ and $f$, and a cycle containing $f$ and $g$, then there is a cycle containing $e$ and $g$.

**2.** Let $d_1, d_2, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

**3.** Let $T$ be a tree, and let $T_1, \ldots, T_n$ be connected subgraphs of $T$ so that $V(T_i \cap T_j) \neq \emptyset$ for all $i, j$ with $1 \leq i < j \leq n$. Show that $V(T_1 \cap T_2 \cap \ldots \cap T_n) \neq \emptyset$. [*Hint*: Delete a leaf and use induction on $|V(T)|$.]

**4.** Let $v_1, v_2, v_3$ be distinct vertices of a graph $G$ such that $G \setminus v_1$, $G \setminus v_2$, $G \setminus v_3$ are all acyclic. Show that $G$ contains at most one cycle.

**5.** Let $G$ be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there exists a path in $G$ from $u$ to $v$ of length at most $k$. Show that either $G$ contains a cycle of length $\leq 2k + 1$ or $G$ is a tree.