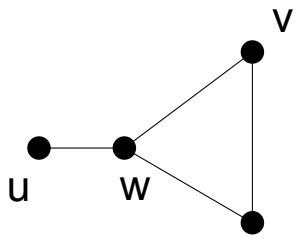

Figure 1: Counterexample for Problem 1a).

MATH 350: Graph Theory and Combinatorics. Fall 2013.

Assignment #1: Paths, Cycles and Trees. Solutions.

1. For each of the following statements decide if it is true or false, and either prove it or give a counterexample.

a) If u, v, w are vertices of G, and there is an even length path from u to v and an even length path from v to w then there is an even length path from u to w.

Solution: False. See Figure 1.

b) If G is connected and has no path with length larger than k, then every two paths in G of length k have at least one vertex in common.

Solution: True. Suppose for a contradiction that P_1 and P_2 are two vertex disjoint paths of length k. Let vertices of P_i be $v_1^i, v_2^i, \ldots, v_{k+1}^i$, in order. Let Q be the a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be a short as possible. Let v_n^1 and v_m^2 be the ends of Q. We can suppose without loss of generality that $m, n \geq k/2 + 1$. Then a path obtained by taking the union of the subpath of P_1 from v_1^1 to v_n^1, the path Q and the subpath of P_2 from v_1^2 to v_m^2 has at least $m + n \geq k + 2$ vertices, a contradiction.

c) If u, v, w are vertices of G, and there is a cycle of G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

Solution: False. Consider a graph G with $V(G) = \{u, v, w\}$ and $E(G)$ consisting of a pair of edges joining u to v and a pair of edges joining v to w.

d) If e, f, g are edges of G, and there is a cycle containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

Solution: True. Without loss of generality we may assume that G is connected. The result follows immediately from the next claim.

Claim: If there exist does not exist a cycle containing edges e and g then there does not exist a vertex $u \in V(G)$ such that every path in G sharing one end with e and another with g contains u.

Proof: The claim trivially holds if e or g is a loop, so we assume that neither is. Let P with vertex set $v_1, v_2, \ldots, v_k$, in order, be a path with e joining v_1 to v_2 and g joining v_{k-1} and v_k. Let $f_i \in E(P_i)$ be the edge with ends v_i and v_{i+1}. Let j be chosen minimum so that no cycle in G contains e and f_j. We will show that $u = v_j$ satisfies the claim.

Suppose not. Let C be a cycle containing e and f_{j-1} and let P' be a path from an end of e to an end of f avoiding u. Choose a subpath Q of P' with one end in $V(C)$ and another in $\{v_{j+1}, v_{j+2}, \ldots, v_k\}$ as short as possible. Then $C \cup Q \cup P$ contains a cycle containing both e and f_j, a contradiction. (The last statement requires some case checking.)

2. Let $d_1, d_2, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

Solution: "Only if" direction: By Theorems 1.1 and 3.1, if T is a tree with degrees $d_1, d_2, \ldots, d_n$ then

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2|V(T)| - 2 = 2n - 2.$$

"If" direction: By induction on n. The base case $n = 2$ is trivial, as K_2 is the unique tree on two vertices. For the induction step, if $n > 2$, then $n < \sum_{i=1}^{n} d_n = 2n - 2 < 2n$, therefore at least one of the d_i's is equal to 1, and at least one of the d_i's is bigger than 1. Without loss of generality, $d_n = 1$, $d_{n-1} > 1$. By the induction hypothesis there exists a tree T' with vertex degrees $d_1, d_2, \ldots, d_{n-1} - 1$. Let T be obtained from T' by adding a leaf to it with the unique neighbor of the leaf being a vertex of degree $d_{n-1} - 1$. It is easy to check that T is a tree and has degrees $d_1, d_2, \ldots, d_n$.

3. Let T be a tree, and let $T_1, \ldots, T_n$ be connected subgraphs of T so that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$. Show that $V(T_1 \cap T_2 \cap \ldots \cap T_n) \neq \emptyset$.

Solution: Proof by induction on $V(T)$. Base case $|V(T)| = 1$ is trivial. For the induction step, let v be a leaf of T and let u be the unique vertex of T adjacent to v. Let $T' = T \setminus v$ and let $T'_i = T' \setminus v$ for $i = 1, 2, \ldots, n$. If $V(T'_i \cap T'_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$, then we can apply the induction hypothesis to T' to complete the proof. Thus we may assume, without loss of generality, that $V(T'_1) \cap V(T'_2) = \emptyset$. It follows that $V(T_1) \cap V(T_2) = \{v\}$. Thus either $u \notin V(T_1)$ or $u \notin V(T_2)$. Without loss of generality, we have $V(T_1) = \{v\}$. Therefore $v \in V(T_i)$ for every $1 \leq i \leq n$ by the assumption and $v \in V(T_1 \cap T_2 \cap \ldots \cap T_n)$, as desired.

4. Let v_1, v_2, v_3 be distinct vertices of a graph G such that $G \setminus v_1$, $G \setminus v_2$, $G \setminus v_3$ are all acyclic. Show that G contains at most one cycle.
Solution: Suppose for a contradiction that C_1 and C_2 are two distinct cycles in G. We have $v_1, v_2, v_3 \in V(C_1 \cap C_2)$. Let P be a path with ends in $V(C_1)$ so that $P \subseteq C_2$, $P \subsetneq C_1$, chosen to be as short as possible. (We can choose such a path as a subpath of C_2 with ends v_1 and v_2 satisfies the required conditions.) Then no internal vertex of P belongs to C_1. Let Q_1 and Q_2 be the two paths in C_1 with the same ends as P. Then $C_1 = Q_1 \cup Q_2$ and $C_3 := Q_1 \cup P$ and $C_4 := Q_2 \cup P$ are also cycles. As C_1, C_3 and C_4 have only two vertices in common (the ends of P), one of $G \setminus v_1$, $G \setminus v_2$, $G \setminus v_3$ contains C_1, C_3 or C_4. A contradiction.

5. Let G be a non-null graph such that for every pair of vertices $u, v \in V(G)$ there exists a path in G from u to v of length at most k. Show that either G contains a cycle of length $\leq 2k + 1$ or G is a tree.
Solution: Clearly, G is connected. If G is not a tree then it contains a cycle. Let C be the cycle in G of smallest length and let $v_1, v_2, \ldots, v_l$ be the vertices of C in order. Suppose for a contradiction that $l > 2k+1$. Let P be the shortest path from v_1 to v_{k+1} in G. Then P has length at most k and it follows that $P \subsetneq C$. Thus there exists a subpath Q of P with ends $v_i, v_j \in V(P)$ and otherwise disjoint from C. The union of Q with each of the two paths in C with ends v_i and v_j is a cycle, and so each of these cycles must have length at least l. The sum of their lengths, however, is equal to $l + 2|E(Q)| \leq l + 2|E(P)| \leq l + 2k < 2l$, a contradtiction.