MATH 350: Graph Theory and Combinatorics. Fall 2013.

Assignment #5: Planar graphs. Solutions.

1. A graph G is *outerplanar* if it can be drawn in the plane so that every vertex is incident with the infinite region. Show that a graph G is outerplanar if and only if G has no K_4 or $K_{2,3}$ minor.

Solution: Let G' be obtained from G by adding an extra vertex to G adjacent to every other vertex. Then G' is planar if and only if G is outerplanar and it is easy to cherk that G contains a K_4 or $K_{2,3}$ minor if and only if G' contains a K_5 or $K_{3,3}$ minor. Thus the problem follows from Kuratowski's theorem (18.2).

2. Let G be a simple 2-connected graph drawn in the plane so that all vertices are incident with the infinite region. Suppose that every bounded region of G has length 3. Let k be the number of vertices of degree 2 in G, and let r be the number of regions of G sharing no edges with the infinite region. Show that $k = r + 2$ if $|V(G)| > 3$.

Solution: Let $|V(G)| = n$ and let t be the total number of bounded regions of G. Note that a bounded region of G shares two edges with the infinite region if and only if one of the vertices on the boundary of the region has degree 2. The infinite region of G is bounded by a cycle of length n and thus $n = 2k + (t - k - r) = t + k - r$. On the other hand, $2|E(G)| = 3t + n$ and by Euler's formula we have $n - (3t + n)/2 + (t + 1) = 2$, or equivalently $n = t + 2$. Combining these two identities we obtain $t + k - r = t + 2$, implying $k = r + 2$.

3. Let G be a 2-connected loopless graph drawn in the plane. For each vertex v, define $S(v) = \frac{1}{2} - \frac{1}{deg(v)}$. Show that for some region R, $\sum_{v \sim R} S(v) < 1$, where the sum is over all vertices v incident with R.

Solution: Suppose that

$$\sum_{v \sim R} S(v) \geq 1 \tag{1}$$

for every region R. Note that every vertex $v \in V(G)$ belongs to exactly

$\deg(v)$ regions. Let r denote the number of regions of the drawing of G. Summing (1) over all the regions we obtain

$$\begin{aligned} r &\leq \sum_R \sum_{v \sim R} S(v) = \sum_{v \in V(G)} \deg(v) S(v) \\ &= \sum_{v \in V(G)} \frac{1}{2} \deg(v) - \sum_{v \in V(G)} 1 = |E(G)| - |V(G)|. \end{aligned}$$

It follows that $|V(G)| - |E(G)| + r \leq 0$, contradicting the Euler's formula.

4. Let G be a graph drawn in the plane. Show that G is bipartite if and only if every region of the drawing has even length.

Solution: Every region of G is bounded by a union of vertex disjoint closed walks. If G is bipartite then every closed walk has even length and so every region has an even length.
We now show that if every region of the drawing has even length then G is bipartite. The proof is by induction on $|E(G)|$. The base case ($|E(G)| = 0$) is trivial. For the induction step, we may assume that G is not a forest. Let e be an edge of G which is not a cut-edge and let u and v be its ends. Thus the regions on the two sides of e are distinct. Let R be a region on one of the side of e. There exists a closed walk W' in G which contains e only once and is part of the boundary of R. Let W be a walk from u to v obtained from W' by removing e. By the induction hypothesis,the graph $G \setminus e$ is bipartite, and the walk W has odd length. It follows that u and v belong to the different parts of every bipartition of $G \setminus e$ and thus G is bipartite.

5. Let G be drawn in the plane so that

- the boundary of the infinite region is some cycle C,
- every other region has boundary a cycle of length 3,
- every vertex of G not in C has even degree.

Show that $\chi(G) \leq 3$.

Solution: Following the hint, we prove the result by induction on $|V(G)|$. The induction base $|V(G)| = 3$ is trivial. For the induction step, suppose first that some two vertices of C are joined by an edge $e \notin E(C)$. We can express $C + e$ as a union of two cycles C_1 and C_2. Let G_1 and G_2 be the subgraphs of G bounded by C_1 and C_2, respectively. Then G_1 and G_2 are 3-colorable by the induction hypothesis and we can combine their colorings to produce a coloring of G, as $G_1 \cap G_2$ consists of a pair of adjacent vertices.

Suppose now that G contains no edge e as above. Consider $v \notin V(C)$. Assume first that there are no parallel edges incident to v. Let $u_0, u_1, \dots, u_k$ be the neighbors of v, listed in the order that the edges incident to v appear around it, with $u_0, u_k \in V(C)$. In a graph $G \backslash v$ the infinite face is bounded by a cycle obtained from C by replacing v with $u_0, u_1, \dots, u_k$. By the induction hypothesis there exists a 3-coloring $c : V(G \backslash v) \to \{1, 2, 3\}$. Note that for each vertex $u \in V(G \backslash v)$ the colors of its neighbors alternate as we enumerate these neighbors in the order edges incident to u appear around u. As $\deg(u_i)$ is even for $i = 1, 2, \dots, k-1$ we deduce that $c(u_{i-1}) = c(u_{i+1})$ for each such i. It follows that only two colors are used on $u_0, u_1, \dots, u_k$ and thus c can be extended to v.

Suppose, finally, that some vertex $v \in V(C)$ is joined to another vertex $u \in V(G)$ by a pair of parallel edges e and f. (The graph G does not contain any loops as every finite face is bounded by a cycle of length 3.) Let R be the region of the plane bounded by e and f. Let G_1 be the subgraph of G drawn within R. Let G_2 be obtained from G by deleting the vertices within the interior of R and the edge f. Applying the induction hypothesis to G_1 and G_2 we obtain a valid coloring of G, as in the previous paragraph. The only caveat is that we need to verify that the degree of u is even in G_2.

Suppose not. Then the degree of u is even in G_1. Consider the graph G_1^* dual to G_1. In G_1^* every region is bounded by an even cycle. Thus G_1^* is bipartite, but every vertex of G_1^* has degree 3, except for one vertex of degree 2, corresponding to the infinite face of G_1. Summing degrees of the vertices on either side of the bipartition we obtain is contradiction, as one of the sums will be divisible by 3, but not the other. This contradiction finishes the proof.