MATH 350: Graph Theory and Combinatorics. Fall 2013.

Assignment #4: Matchings and coloring. Solutions.

1. Let G be a graph and $Z \subseteq V(G)$. Show that the following are equivalent:

(i) G has a matching covering Z, and

(ii) for every $X \subseteq V(G)$ there are at most $|X|$ odd components C of $G \backslash X$ such that $V(C) \subseteq Z$.

Solution: (i) $\Rightarrow$ (ii): Let M be a matching covering Z. For every $X \subseteq V(G)$ and every component C of $G \backslash X$, some edge of M joins a vertex of X to a vertex of C. Thus there at least as many vertices as there are such odd components.
(ii) $\Rightarrow$ (i): Let G' be obtained from G by adding a set Y of $|V(G)|$ extra vertices, so that the vertices of Y are pairwise adjacent and every vertex of Y is adjacent to every vertex in $V(G) - Z$. Any matching in G covering Z can be extended to a perfect matching in G'. Conversely, every perfect matching of G' contains a matching in G covering Z. Thus by Tutte's theorem (13.1) (i) holds as long as the following condition holds:

(iii) $G' \backslash X'$ has at most $|X'|$ odd components for every $X' \subseteq V(G')$.

Condition (iii) is trivially satisfied when $Y \subseteq X'$ and we assume that $Y \not\subseteq X'$. All the vertices of $V(G') - Z - X'$ belong to the same component of $G' \backslash X'$. Let $X = X' \cap V(G)$. By (ii) and the preceding observation $G' \backslash X'$ has at most $|X| + 1$ odd components. As in the proof of Theorem 13.1 the number of odd components of $G' \backslash X'$ has the same parity as $|X'|$. Therefore (iii) holds, as desired.

2. Let G be a 2-connected 3-regular graph and let e be an edge of G. Show that e belongs to some perfect matching of G.

Solution: Let u, v be the ends of e. Our goal is to show that the graph $G' := G \setminus u \setminus v$ contains a perfect matching. Suppose not, then by Tutte's theorem there exists $X \subseteq V(G')$ such that $G' \setminus X (= G \setminus (X \cup \{u, v\}))$ has $> |X|$ odd components, which we denote by $C_1, C_2, \ldots, C_l$. Let $k := |X|$. As $|V(G)|$ is even, we have $l \geq k + 2$. Let F denote the set of all edges with one end in $X \cup \{u, v\}$ and another in $V(G) - X - \{u, v\}$. Then every C_i is incident with ≥ 3 edges of F, by our assumption on the connectivity of G, and so $|F| \geq 3l \geq 3k + 6$. On the other hand, $|F| \leq 3|X| + 4 = 3k + 4$, yielding a contradiction.

3. Show that if G is a loopless graph, $k \geq 1$ is an integer and $\chi(G) > k$ then G has a path with k edges.

Solution: Order the vertices of G arbitrarily: $v_1, v_2, \ldots, v_n$. We apply the coloring algorithm with respect to this ordering. If v_i receives color c then it has a neighbor v_j with $j < i$ such that v_j receives color $c - 1$. This observation easily implies by induction on c that every vertex v_i which receives color c is an end of some path P with $c - 1$ edges such that $V(P) \subseteq \{v_1, v_2, \ldots, v_i\}$. Thus, if a vertex receives a color $k+1$ then it is an end of a path with k edges, as desired.

4. Let G be a loopless graph in which every two odd cycles share a vertex. Show that $\chi(G) \leq 5$.

Solution: If G is bipartite then the result clearly holds. Otherwise, by 7.1 there exists an induced odd cycle C in G. Then $G \setminus V(C)$ is bipartite and can be colored in 2 colors, while C can be colored in 3.

5. Let G be a loopless graph with $\chi(G) = k$ for some positive integer k. Show that G contains at least k vertices with degree $\geq k-1$.

Solution: Order the vertices of G in the non-increasing order of degrees and apply the coloring algorithm to this ordering. If a vertex v receives color k then it must be preceded by $\geq k-1$ of its neighbors in the ordering. Thus v has degree $\geq k-1$ and so does each of its $\geq k-1$ neighbors preceding it in the ordering