MATH 350: Graph Theory and Combinatorics. Fall 2013.

Assignment #3: Network flows, covers and Ramsey's theorem.

1. Let G be a directed graph and for each edge e let $\phi(e) \geq 0$ be an integer, so that for every vertex v,

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

Show there is a list $C_1, ..., C_n$ of directed cycles (possibly with repetition) so that for every edge e of G,

$$|\{i : 1 \leq i \leq n,\ e \in E(C_i)\}| = \phi(e).$$

Solution: Induction on $S := \sum_{e \in E(G)} \phi(e)$. Base case: $S = 0$ is trivial. For the induction step, it suffices to find a directed cycle C in G so that $\phi(e) \geq 1$ for every edge $e \in E(G)$, as one can then apply the induction hypothesis to

$$\phi'(e) := \begin{cases} \phi(e), & \text{if } e \notin E(G) \\ \phi(e) - 1, & \text{if } e \in E(G) \end{cases}$$

Let e be an edge of G with $\phi(e) \geq 1$, a tail u and a head v. Then ϕ restricted to $G \setminus e$ is a v-u-flow of value 1 and by Lemma 10.3 there exists a directed path P in $G \setminus e$ so that ϕ is positive on every edge of the path. The path P together with e forms the desired cycle.

2. Let s, t be vertices of a digraph G, and let $\phi : E(G) \to \mathbb{R}_+$ be an $s - t$ flow. Show that there is an $s - t$ flow $\psi : E(G) \to \mathbb{Z}_+$ so that

(i) its total value is at least that of ϕ, and

(ii) $|\psi(e) - \phi(e)| < 1$ for every edge e of G.

The proof is by induction on

$$S(\phi) := \max(\lceil \text{value}(\phi) \rceil, 0) + 2 \sum_{e \in E(G)} \lfloor \phi(e) \rfloor.$$

The base case: If $S(\phi) = 0$ then $\psi \equiv 0$ satisfies the required conditions.
Induction step: Suppose first that value$(\phi) > 0$. Then there exists a directed $s-t$-path P in G such that $\phi(e) > 0$ for every $e \in E(P)$. We define a new flow $\phi' : E(G) \to \mathbb{R}_+$, as follows. Let ϕ' be a flow equal to ϕ on every edge in $E(G) - E(P)$. Let $\phi'(e) = \phi(e) - 1$ for every $e \in E(P)$, where for edges $e \in E(P)$ with $\phi'(e) < 1$ we set $\phi'(e) = 1 - \phi(e)$, instead of $\phi(e) - 1$, and we change the direction of such edges. It is straighforward to check that ϕ' is, indeed, a flow. Further, value$(\phi') =$ value$(\phi) - 1$, and $\lfloor \phi'(e) \rfloor \leq \lfloor \phi(e) \rfloor$ for every $e \in E(G)$. It follows that $S(\phi') < S(\phi)$, and there exists a flow $\psi' : E(G) \to \mathbb{Z}_+$ satisfying the induction hypothesis for ϕ'.
Let ψ be defined to be identical to ψ' on $E(G) - E(P)$ and let $\psi(e) = \psi'(e) + 1$, for $e \in E(P)$. Once again we need to be careful with the edges for which the direction has changed between ϕ and ϕ': If $\phi(e) < 1$ for some $e \in E(P)$, and $\psi'(e) = 1$ for the edge e with the reverse direction then we set $\psi(e) = 0$. One can verify that value$(\psi) =$ value$(\psi') + 1$ and that $|\psi(e) - \phi(e)| = |\psi'(e) - \phi'(e)|$. It follows that ψ satisfies conditions (i) and (ii), as desired.
It remains to consider the case when value$(\phi) \leq 0$. An argument analogous to the one used in the previous case can produce an appropriate flow ψ using a directed $t-s$-path P' or a directed cycle in C in G, instead of the path P, as long as at least one edge e of the corresponding path or cycle satisfies $\phi(e) \geq 1$ and all the remaining edges have positive value of the flow.
Let $e \in E(G)$ directed from u to v satisfy $\phi(e) \geq 1$. (If no such edge exists then $S(\phi) = 0$ and the base case applies.) Consider the graph G' containing only the edges of G with the positive flow value. If G' contains a directed path from v to u then G contains a cycle as described in the preceding paragraph and the proof is finished. Otherwise, there exists $X \subseteq V(G)$ with $v \in X$, $u \in V(G) - X$ such that every edge of G' with one end in X and another in $V(G) - X$ is directed from $V(G) - X$ to X. As value$(\phi) \leq 0$, it follows that $s \in X, t \in V(G) - X$. One can now show that there exists a directed path from v to s in X and from t to u in $V(G) - X$. Thus e belongs to a directed $t-s$-path P' in G' and as mentioned in te previous paragraph the argument analogous to the one used for a directed $s-t$-path completes the proof.

3. Show that $R(3,4) = 9$.

Solution: We start by showing that $R(3,4) \leq 9$. Suppose for a contradiction that G is a graph on ≥ 9 vertices containing no independent set of size 3 and no clique of size 4. By Theorem 11.1 $R(3,3) \leq R(3,2) + R(2,3) = 6$. If $\deg(v) \geq 6$ for some vertex $v \in V(G)$, then as in the proof of Theorem 11.1 we obtain a contradiction. Similarly, as $R(2,4) = 4$ no vertex of G can have 4 non-neighbors. We deduce that $|V(G)| = 9$ and $\deg(v) = 5$ for every $v \in V(G)$. But this is impossible, as $\sum_{v\ inV(G)} \deg(v)$ must be even.

To show that $R(3,4) > 8$ consider a graph G with $V(G) = \{1, 2, \dots, 8\}$ with the vertices i and j being adjacent in G if and only if $|i-j| \not\in \{1, 4, 7\}$. It is not hard to check that G contains neither an independent set of size 3, nor a clique of size 4.

4. Show that any coloring of edges of K_n with $n \geq 6$ in two colors contains at least $\frac{1}{20}\binom{n}{3}$ monochromatic triangles.

Solution: We know that every coloring of edges of K_6 contains a monochromatic triangle. There are $\binom{n}{6}$ ways of choosing 6 vertices out of n and every such choice yields a monochromatic triangle, while every monochromatic triangle can be counted up to $\binom{n-3}{3}$ times. Thus there must be at least $\binom{n}{6} / \binom{n-3}{3} = \frac{1}{20}\binom{n}{3}$ monochromatic triangles.

5. Let

$$r_k := R_k(\underbrace{3, 3, \dots, 3}_{k \text{ times}}).$$

(I.e. r_k is the minimum integer $n > 0$ such that every coloring of edges of K_n in k colors is guaranteed to produce a monochromatic triangle.) Show that

$$r_k \leq k(r_{k-1} - 1) + 2$$

for $k \geq 2$.

Solution: We need to show that for $n = k(r_{k-1} - 1) + 2$ every coloring of edges of K_n in k colors is guaranteed to produce a monochromatic triangle. Let v be an arbitrary vertex of K_n at least $\lceil \frac{n-1}{k} \rceil = r_{k-1}$ edges incident with v have the same color, say k. Let X be the set of vertices joined to v by edges of this color. If two vertices of X are joined by an edge of color k

they form a monochromatic triangle with v. Otherwise, only $k-1$ colors are used on edges joining the vertices of X, and the subgraph induced on X contains a monochromatic triangle by definition of r_{k-1}.