1. Show that every loopless graph $G$ contains a bipartite subgraph with at least $|E(G)|/2$ edges.

Solution: Let a partition $(A, B)$ of $V(G)$ be chosen so that the number of edges of $G$ with one end in $A$ and another in $B$ is maximized. Let $F$ be the set of these edges, and let $H$ be the bipartite subgraph of $G$ with $V(H) = V(G)$ and $E(H) = F$. It suffices to show that $\deg_H(v) \geq \frac{1}{2} \deg_G(v)$ for every $v \in V(G)$. Suppose not and for some $v \in V(G)$ we have $\deg_H(v) < \frac{1}{2} \deg_G(v)$. Without loss of generality, $v \in A$. It follows that more edges incident with $v$ have the second end in $A$ then in $B$. Thus the partition $(A - \{v\}, B \cup \{v\})$ has more edges with ends in different parts, contradicting our choice of $(A, B)$.

2. Let $G$ be a loopless graph in which every vertex has degree $\geq 1$. Let $X$ be the largest matching in $G$, and let $Y$ be the smallest set of edges of $G$ so that every vertex of $G$ is incident with $\geq 1$ edge in $Y$. Show that $|X| + |Y| = |V(G)|$.

Solution: See Theorem 11.2 from the Fall 2012 Notes.

3. Let $G$ be a bipartite graph with bipartition $(A, B)$ in which every vertex has degree $\geq 1$. Assume that for every edge of $G$ with ends $a \in A$ and $b \in B$ we have $\deg(a) \geq \deg(b)$. Show that there exists a matching in $G$ covering $A$.

Solution: Suppose not. By Hall’s theorem there exists $X \subset A$ with $< |X|$ neighbors in $B$. Choose such a set $X$ with $|X|$ minimum. Let $Y \subset B$ denote the set of vertices adjacent to any of the vertices in $X$. Then there exists a matching $M$ consisting of edges joining vertices of $Y$ to vertices of $X$ of size $|Y|$. Indeed, otherwise, by Hall’s theorem, there exists $Y' \subset Y$ so that the set $X'$ of vertices in $X$ adjacent to any of the vertices of $Y'$ satisfies $|X'| < |Y'|$. It follows that $|X - X'| < |Y - Y'|$ and, as the vertices of $X - X'$ have no neighbors in $Y'$, we deduce that $X - X'$ contradicts the
minimality of $X$.

Let $F$ denote the set of edges joining $X$ and $Y$. Then

$$|F| = \sum_{a \in A} \deg(a) > \sum_{e = ab \in M} \deg(a) \geq \sum_{e = ab \in M} \deg(b) \geq |F|,$$

a contradiction. Thus $G$ contains a matching covering $A$, as desired.

4. Given integers $n \geq m \geq k \geq 0$, determine the maximum possible number of edges in a simple bipartite graph $G$ with bipartition $(A, B)$, with $|A| = n, |B| = m$ and no matching of size $k$.

Solution: If $G$ has no matching of size $k$ then by König’s theorem it contains a set $X$ with $|X| \leq k - 1$ so that every edge has an end in $X$. Every vertex in $X$ is incident with at most $n$ edges. Therefore, $|E(G)| \leq (k - 1)n$. One can have a graph with these many edges satisfying all the criteria by having exactly $k - 1$ vertices of $B$ with non-zero degree, each joined to all the vertices of $A$.

5. Let $v$ be a vertex in a 2-connected graph $G$. Show that $v$ has a neighbor $u$ such that $G \setminus u \setminus v$ is connected.

Solution: Let $U$ be the set of neighbors of $v$ in $G$. Let $T$ be the minimum connected subgraph of $G \setminus v$ such that $U \subseteq V(T)$. It is easy to see that $T$ is a tree and that every leaf of $T$ is a neighbor of $v$. Let $u$ be a leaf of $T$. Then $T \setminus u$ is connected. Suppose for a contradiction that $G \setminus u \setminus v$ is not connected and consider a component $C$ of $G \setminus u \setminus v$ which does not contain $T \setminus u$. Thus $C$ contains no neighbor of $v$ and so it is a connected component of $G \setminus u$. It follows that $G \setminus u$ is not connected, contradicting 2-connectivity of $G$.

6.

a) Distinct $u, v \in V(G)$ are $k$-linked if there are $k$ paths $P_1, \ldots, P_k$ of $G$ from $u$ to $v$ so that $E(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose $u, v, w$ are distinct and $u, v$ are $k$-linked, and so are $v, w$. Does it follow that $u, w$ are $k$-linked?
**Figure 1**: Counterexample for Problem 6b).

**Solution**: Yes. By Theorem 9.4 if $u$ and $w$ are not $k$-linked then there exists $X \subseteq V(G)$ with $u \in X$, $w \notin X$ and $|\delta(X)| < k$. By symmetry, we may assume $v \in X$. Then the opposite direction of Theorem 9.4 implies that $v$ and $w$ are not $k$-linked.

**b)** Subsets $X, Y \subseteq V(G)$ are $k$-joined if $|X| = |Y| = k$ and there are $k$ paths $P_1, \ldots, P_k$ of $G$ from $X$ to $Y$ so that $V(P_i \cap P_j) = \emptyset$ ($1 \leq i < j \leq k$). Suppose $X, Y, Z \subseteq V(G)$ and $X, Y$ are $k$-joined, and so are $Y, Z$. Does it follow that $X, Z$ are $k$-joined?

**Solution**: No. See Figure 1 for an example with $k = 2$. 