MATH 350: Graph Theory and Combinatorics. Fall 2013.

Assignment #2: Bipartite graphs, matching and connectivity.

1. Show that every loopless graph G contains a bipartite subgraph with at least $|E(G)|/2$ edges.

Solution: Let a partition (A, B) of $V(G)$ be chosen so that the number of edges of G with one end in A and another in B is maximized. Let F be the set of these edges, and let H be the bipartite subgraph of G with $V(H) = V(G)$ and $E(H) = F$. It suffices to show that $\deg_H(v) \geq \frac{1}{2}\deg_G(v)$ for every $v \in V(G)$. Suppose not and for some $v \in V(G)$ we have $\deg_H(v) < \frac{1}{2}\deg_G(v)$. Without loss of generality, $v \in A$. It follows that more edges incident with v have the second end in A then in B. Thus the partition $(A - \{v\}, B \cup \{v\})$ has more edges with ends in different parts, contradicting our choice of (A, B).

2. Let G be a loopless graph in which every vertex has degree ≥ 1. Let X be the largest matching in G, and let Y be the smallest set of edges of G so that every vertex of G is incident with ≥ 1 edge in Y. Show that $|X| + |Y| = |V(G)|$.

Solution: See Theorem 11.2 from the Fall 2012 Notes.

3. Let G be a bipartite graph with bipartition (A, B) in which every vertex has degree ≥ 1. Assume that for every edge of G with ends $a \in A$ and $b \in B$ we have $\deg(a) \geq \deg(b)$. Show that there exists a matching in G covering A.

Solution: Suppose not. By Hall's theorem there exists $X \subset A$ with $< |X|$ neighbors in B. Choose such a set X with $|X|$ minimum. Let $Y \subset B$ denote the set of vertices adjacent to any of the vertices in X. Then there exists a matching M consisting of edges joining vertices of Y to vertices of X of size $|Y|$. Indeed, otherwise, by Hall's theorem, there exists $Y' \subset Y$ so that the set X' of vertices in X adjacent to any of the vertices of Y' satisfies $|X'| < |Y'|$. It follows that $|X - X'| < |Y - Y'|$ and, as the vertices of $X - X'$ have no neighbors in Y', we deduce that $X - X'$ contradicts the

minimality of X.
Let F denote the set of edges joining X and Y. Then

$$|F| = \sum_{a \in A} \deg(a) > \sum_{\substack{e=ab \in M \\ a \in A, b \in B}} \deg(a) \geq$$
$$= \sum_{\substack{e=ab \in M \\ a \in A, b \in B}} \deg(b) \geq |F|,$$

a contradiction. Thus G contains a matching covering A, as desired.

4. Given integers $n \geq m \geq k \geq 0$, determine the maximum possible number of edges in a simple bipartite graph G with bipartition (A, B), with $|A| = n, |B| = m$ and no matching of size k.

Solution: If G has no matching of size k then by König's theorem it contains a set X with $|X| \leq k-1$ so that every edge has an end in X. Every vertex in X is incident with at most n edges. Therefore, $|E(G)| \leq (k-1)n$. One can have a graph with these many edges satisfying all the criteria by having exactly $k-1$ vertices of B with non-zero degree, each joined to all the vertices of A.

5. Let v be a vertex in a 2-connected graph G. Show that v has a neighbor u such that $G \setminus u \setminus v$ is connected.

Solution: Let U be the set of neighbors of v in G. Let T be the minimum connected subgraph of $G \setminus v$ such that $U \subseteq V(T)$. It is easy to see that T is a tree and that every leaf of T is a neighbor of v. Let u be a leaf of T. Then $T \setminus u$ is connected. Suppose for a contradiction that $G \setminus u \setminus v$ is not connected and consider a component C of $G \setminus u \setminus v$ which does not contain $T \setminus u$. Thus C contains no neighbor of v and so it is a connected component of $G \setminus u$. It follows that $G \setminus u$ is not connected, contradicting 2-connectivity of G.

6.

a) Distinct $u, v \in V(G)$ are *k-linked* if there are k paths $P_1, ..., P_k$ of G from u to v so that $E(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose u, v, w are distinct and u, v are k-linked, and so are v, w. Does it follow that u, w are k-linked?

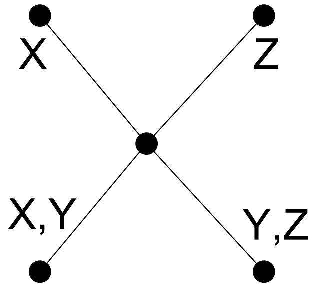

Figure 1: Counterexample for Problem 6b).

Solution: Yes. By Theorem 9.4 if u and w are not k-linked then there exists $X \subseteq V(G)$ with u in X, $w \notin X$ and $|\delta(X)| < k$. By symmetry, we may assume $v \in X$. Then the opposite direction of Theorem 9.4 implies that v and w are not k-linked.

b) Subsets $X, Y \subseteq V(G)$ are *k-joined* if $|X| = |Y| = k$ and there are k paths $P_1, ..., P_k$ of G from X to Y so that $V(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose $X, Y, Z \subseteq V(G)$ and X, Y are k-joined, and so are Y, Z. Does it follow that X, Z are k-joined?

Solution: No. See Figure 1 for an example with $k = 2$.