MATH 340: Discrete Structures II. Winter 2017.

Assignment #6: Generating functions. Solutions.

1. *Fruit salad.* Let $s(n)$ be the number of ways to make a fruit salad with n pieces of fruit, given that we must use strawberries by the half-dozen, an even number of apples, at most five bananas, and at most one pineapple.

a) Evaluate the ordinary generating function for s.

b) Use this to find $s(n)$.

Solution: a): Let $S_s(x), S_a(x), S_b(x), S_p(x)$ be the generating functions corresponding to the number of ways one can make the fruit salad using only strawberries, only apples, only bananas, and only pineapples respectively. Then

$$\begin{aligned} S_s(x) &= 1 + x^6 + x^{12} + \ldots = \frac{1}{1-x^6}, \\ S_a(x) &= 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}, \\ S_b(x) &= 1 + x + x^2 \ldots + x^5 = \frac{1-x^6}{1-x}, \\ S_p(x) &= 1 + x. \end{aligned}$$

Let $S(x)$ be the generating function for x, then

$$S(x) = S_s(x)S_a(x)S_b(x)S_p(x) = \frac{(1-x^6)(1+x)}{(1-x^6)(1-x^2)(1-x)} = \frac{1}{(1-x)^2}$$

b):

$$\sum_{n\geq 0} s(n)x^n = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n$$

Thus $s(n) = n + 1$.

2. *The Round table.* Let $r(n)$ be the number of different ways to seat n people around a round table. Find the exponential generating function for r.

Solution: Number the people $1, 2, \ldots, n$. Let the people sitting to the right of person number n have numbers $x_1, x_2, \ldots, x_{n-1}$ in order. The sequence $x_1, x_2, \ldots, x_{n-1}$ uniquely determines the sitting arrangement, and each such sequence is just a permutation of $[n-1]$. Thus $r(n) = (n-1)!$ and

$$\hat{R}(x) = \sum_{n\geq 0} r(n)\frac{x^n}{n!} = \sum_{n\geq 0} \frac{(n-1)!}{n!}x^n = \sum_{n\geq 1} \frac{x^n}{n} = -\ln(1-x),$$

is the exponential generating function for r. (Assuming $r(0) = 0$.)

3. *Sum of cubes.*
Use generating functions to evaluate

$$\sum_{k=0}^{n} (k-1)k(k+1)$$

.

Solution: Let us start by finding the generating function

$$C(x) = \sum_{n\geq 0} (n-1)n(n+1)x^n.$$

For any generating function $F(x)$ we have

$$x\frac{d}{dx}F(x) = \sum_{n\geq 0} nf(n)x^n.$$

Thus

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \sum_{n\geq 0} (n+1)x^n, \\
\frac{2x}{(1-x)^3} &= \sum_{n\geq 0} n(n+1)x^n, \\
\frac{6}{(1-x)^2} - \frac{12}{(1-x)^3} + \frac{6}{(1-x)^4} &= \sum_{n\geq 0} (n-1)n(n+1)x^n.
\end{aligned}$$

Let $s(n) = \sum_{k=0}^{n}(k-1)k(k+1)$ and let $S(x) = \sum_{n\geq 0} s(n)x^n$. Then

$$\begin{aligned} S(x) &= \frac{C(x)}{1-x} = \\ &= \frac{6}{(1-x)^3} - \frac{12}{(1-x)^4} + \frac{6}{(1-x)^5} \end{aligned}$$

As shown in class, we have

$$\frac{1}{(1-x)^d} = \sum_{n\geq 0} \binom{n+d-1}{d} x^n.$$

Thus

$$\begin{aligned} S(x) &= \sum_{n\geq 0} \left(6\binom{n+2}{2} - 12\binom{n+3}{3} + 6\binom{n+4}{4} \right) x^n \\ &= \frac{1}{4} \sum_{n\geq 0} (n-1)n(n+1)(n+2)x^n. \end{aligned}$$

Finally, we have

$$\sum_{k=0}^{n}(k-1)k(k+1) = \frac{(n-1)n(n+1)(n+2)}{4}.$$

4. *Alternating Permutations.* A permutation $\pi_1, \pi_2, \ldots, \pi_n$ of numbers $1, 2, \ldots, n$ is *alternating* if

$$\pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots.$$

Let $a(n)$ be the number of alternating permutations of size n.

a) Find a recurrence relation for $a(n)$.

b) Evaluate the exponential generating function for a.

Solution: a): Given an alternating permutation of $[n+1]$ with $\pi_{k+1} = n+1$, we have

- $k+1$ is odd,
- the numbers $\pi_1, \pi_2, \ldots, \pi_k$ form an alternating permutation,

- the numbers $-\pi_{k+2}, -\pi_{k+3}, \ldots, -\pi_n$ form an alternating permutation.

Thus

$$a(n+1) = \sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k} a(k)a(n-k).$$

To simplify the recursion let us also consider the permutations with $\pi_{k+1} = 1$. Then

- $k+1$ is even
- the numbers $\pi_1, \pi_2, \ldots, \pi_k$ form an alternating permutation,
- the numbers $\pi_{k+2}, \pi_{k+3}, \ldots, \pi_n$ form an alternating permutation.

Thus

$$a(n+1) = \sum_{\substack{1 \le k \le n \\ k \text{ is odd}}} \binom{n}{k} a(k)a(n-k),$$

and, summing the two expressions, we have

$$2a(n+1) = \sum_{k=0}^{n} \binom{n}{k} a(k)a(n-k).$$

This holds for $n \ge 1$. Moreover, $a(0) = a(1) = 1$.

b): Let $\hat{A}(x) = \sum_{n\ge 0} a^n \frac{x^n}{n!}$. By the formula for the product of exponential generating functions

$$\begin{aligned}\hat{A}^2(x) &= \sum_{n\ge 0}\left(\sum_{k=0}^{n}\binom{n}{k}a(k)a(n-k)\right)\frac{x^n}{n!} \\ &= 1 + 2\sum_{n\ge 1} a(n+1)\frac{x^n}{n!} = -1 + 2\sum_{n\ge 0} a(n+1)\frac{x^n}{n!} = 2\hat{A}'(x) - 1.\end{aligned}$$

Therefore,

$$2\hat{A}'(x) = \hat{A}^2(x) + 1,$$

$$\int \frac{\hat{A}'(x)}{\hat{A}^2(x)+1} = x + C$$

$$2\tan^{-1}(\hat{A}(x)) = x + C$$

$$\hat{A}(x) = \tan\left(\frac{x + C}{2}\right).$$

As $\hat{A}(0) = a(0) = 1$, we have $C = \pi/2$, and

$$\hat{A}(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) (= \sec x + \tan x).$$