MATH 340: Discrete Structures II. Winter 2017.
Due in class on Wednesday, March 22nd.

Assignment #4: Discrete Probability II.

1. *The Birthday problem.* Suppose that the birthdays of n people in the room are uniformly distributed among the 365 days of the year. Estimate how large should n be to guarantee that the probability of some two people sharing a birthday is at least 999/1000.

2. *Quicksort.* Let $x_1, x_2, \ldots, x_n$ be a permutation of numbers $1, \ldots, n$ chosen uniformly at random.

a) Show that the probability that the numbers i and j, such that $1 \leq i \leq j \leq n$, are compared to each other by the Quicksort algorithm is equal to

$$\frac{2}{j-i+1}.$$

b) Deduce that the expected number of comparisons made by the Quicksort algorithm is equal to

$$2\sum_{k=1}^{n-1} \frac{n-k}{k+1}.$$

3. *Balls and bins.* Suppose that we randomly drop $n^{3/2}$ balls into n bins. Give an upper bound on the expectation of the maximum number of balls in any bin.

4. *Balls and bins II.* Given n balls of each of n different colors (n^2 balls in total), we distribute them among n boxes, as follows. For each ball we choose a box at random. If the chosen box already contains the ball of the same color as the ball we are considering, we throw the current ball away. Otherwise, we put it in the box.

a) Show that the probability that a box contains a ball of given color is

$$1 - \left(1 - \frac{1}{n}\right)^n.$$

b) Find the expected number of balls that we throw away.

c) Show that with high probability no box contains more than

$$n\left(1 - \left(1 - \frac{1}{n}\right)^n\right) + 2\sqrt{n \ln n}$$

balls.

5. *Deviation below the mean.* Prove the following variant of the Chernoff bound for the deviation below the mean. Let X be the sum of independent Bernoulli random variables, and let $\mu = \mathrm{E}[X]$. Show that, if $0 < \delta < 1$, then

$$p(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^\mu \leq e^{-\mu\delta^2/2}$$

6. *Random graphs.* In a *random graph* on n vertices for each pair of vertices i and j we independently include the edge $\{i, j\}$ in the graph with probability $1/2$. Show that with high probability every two vertices have at least $n/4 - \sqrt{n \log n}$ common neighbors.