MATH 340: Discrete Structures II. Winter 2017.

Assignment #2: Planar graphs. Solutions.

1. *Euler's formula.*

a) Let G be a planar graph, such that every vertex of G has degree at least five, and at least one vertex of G has degree eight. Show that G has at least fifteen vertices.

b) Let G be a triangulation of the plane. Show that the number of faces of G is even.

Solution:

a): We have $|E(G)| \leq 3|V(G)| - 6$. As shown in class this formula can be rewritten as

$$\sum_{v \in V(G)} (6 - \deg(v)) \geq 12.$$

Let w be a vertex of degree eight. Then we have

$$\sum_{v \in V(G), v \neq w} (6 - \deg(v)) + (6 - \deg(w)) \geq 12,$$

that is

$$\sum_{v \in V(G), v \neq w} (6 - \deg(v)) \geq 14.$$

As every term in the sum on the left is at most one, there must be at least 14 terms, implying that G has at least fifteen vertices in total.

b): We have seen that

$$\sum_{F} \text{length}(F) = 2|E(G)|,$$

where the sum is taken over all faces F of G. As every face of G has odd length, but the sum of length is even, G must have an even number of faces.

2. *Coloring planar graphs.*

a) Show without using the Four Color Theorem that if a planar graph G has no K_3 subgraph then $\chi(G) \leq 4$.

b) Prove or disprove the following statement: If a planar graph G has no K_4 subgraph then $\chi(G) \leq 3$.

Solution:

a): The argument shown in class can be extended from bipartitie graphs to planar graphs with no K_3 subgraph to show that

$$|E(G)| \leq 2|V(G)| - 4$$

for every such planar graph with $|V(G)| \geq 3$. It follows that every planar graph with no K_3 subgraph contains a vertex of degree at most three. (Otherwise, by the handshaking lemma we would have $|E(G)| \geq 2|V(G)|$.) We can now proceed by induction as in several arguments before. The base case ($|V(G)| \leq 2$) is easy. For the induction step, let v be a vertex of G with $\deg(v) \leq 3$. By the induction hypothesis there exists a coloring of $G \setminus v$ in four colors, and it can be extended to v by assigning to it the color not used by its neighbors.

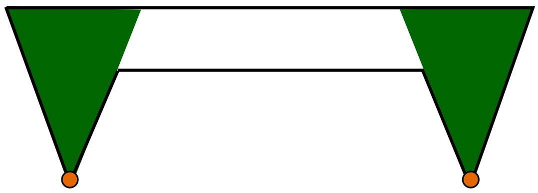
b): The statement is <u>false</u>. Consider for example a graph obtained from the cycle of length five, by drawing a vertex v inside the cycle and joining it by edges to all vertices of the cycle. The resulting graph G is not colorable by three colors, as the color of v can not be used on any other vertex, and the cycle of length five is not bipartite. On the other hand, G has no K_4 subgraph.

3. *Art Gallery theorem.*

a) Let P be a polygon in the plane such that at most two angles of P exceed $180°$. Show that P can be guarded by two guards.

b) Give an example of a polygon as in a) showing that two guards are sometimes required.

Solution:



a): Let u, v be the two vertices of P such that the angles of P at all the other vertices do not exceed $180°$. Let L be a bisector of the angle of P at u, and let u' be the point of intersection of L with the boundary of P closest to u. Then the segment of L joining u and u' divides P into two polygons, say P_1 and P_2. Suppose without loss of generality that v belongs to P_2. Using a bisector at v we can divide P_2 into two polygons Q_1 and Q_2 such that the angle at v in both of them does not exceed $180°$. Thus P_1, Q_1 and Q_2 have no angles exceeding $180°$ and are convex. Therefore each of these polygons can be guarded by one guard placed anywhere within. It follows that the guards placed at u and v can guard P.

b): The polygon is shown on the figure above. It requires two guards, as the vertices indicated by orange circles require a guard in the corresponding green region to guard them.

4. *Kuratowski's theorem.* Let G be a non-planar graph. Suppose further that $G \setminus e$ is planar for every edge e of G. Show that at most six vertices of G have degree three or greater.

Solution: By Kuratowski's theorem, G contains a subgraph H such that H is a subdivision of K_5 or a subdivision of $K_{3,3}$. If $E(G) \neq E(H)$, then there exists an edge $e \in E(G) - E(H)$, and $G \setminus e$ is not planar, in contradiction with our assumption. Therefore $E(G) \neq E(H)$. As H contains at most six vertices of degree at least three, so does G.

5. *Testing planarity.* Determine whether the following two graphs are planar. (For each graph either provide a planar drawing, or prove that this graph is not planar.)

Solution: The left graph is non-planar, as a subdivision of $K_{3,3}$ is indicated in the following figure. The right one is planar, see the planar drawing below.