MATH 340: Discrete Structures II. Winter 2017.

Assignment #1: Matchings.

1. *Stable matching algorithm.* Apply the Boy Proposal algorithm to find a stable matching given the preference lists below. Are there any other stable matchings?

$$\begin{aligned}
\mathbf{B_1} &: G_3 > G_2 > G_1 > G_4 > G_5 \\
\mathbf{B_2} &: G_2 > G_1 > G_3 > G_5 > G_4 \\
\mathbf{B_3} &: G_2 > G_5 > G_4 > G_3 > G_1 \\
\mathbf{B_4} &: G_1 > G_3 > G_4 > G_2 > G_5 \\
\mathbf{B_5} &: G_2 > G_3 > G_1 > G_5 > G_4
\end{aligned}$$

$$\begin{aligned}
\mathbf{G_1} &: B_5 > B_2 > B_1 > B_4 > B_3 \\
\mathbf{G_2} &: B_3 > B_1 > B_4 > B_2 > B_5 \\
\mathbf{G_3} &: B_2 > B_5 > B_4 > B_3 > B_1 \\
\mathbf{G_4} &: B_1 > B_3 > B_4 > B_5 > B_2 \\
\mathbf{G_5} &: B_4 > B_1 > B_5 > B_3 > B_2
\end{aligned}$$

Solution: In the Boy Proposal Algorithm there is a potential choice involved in each step: Which one of the currently unengaged boys proposes next. We will always choose the one with the lowest number. We will record a proposal of B_i to G_j in the form B_iG_j. We will indicate that the proposal is accepted by writing $B_iG_j!$ The algorithm results in the following sequence of proposals:

$$\begin{gathered}
B_1G_3!, B_2G_2!, B_3G_2!, B_2G_1!, B_4G_1, B_4G_3!, B_1G_2, \\
B_1G_1, B_1G_4!, B_5G_2, B_5G_3!, B_4G_4, B_4G_2, B_4G_5!
\end{gathered}$$

with the final matching

$$B_1G_4, B_2G_1, B_3G_2, B_4G_5, B_5G_3.$$

It follows from the results established in class that, if there exists two different stable matchings, then the Girl Proposal Algorithm yields a different stable matching from the Boy Proposal Algorithm. And indeed in the Girl Proposal Algorithm each girl can end up with their top choice:

$$B_1G_4, B_2G_3, B_3G_2, B_4G_5, B_5G_1,$$

which is different from the matching above.

2. *Stable roommates.* We wish to pair up an even number of students in a student dormitory. Each student has a preference list over every other potential roommate. Give an example to show that a stable matching need not exist.

Solution: Here is an example with for students:

$$\begin{aligned}
\mathbf{S_1} &: S_2 > S_3 > S_4 \\
\mathbf{S_2} &: S_3 > S_1 > S_4 \\
\mathbf{S_3} &: S_1 > S_2 > S_4 \\
\mathbf{S_4} &: S_1 > S_2 > S_3
\end{aligned}$$

There are three possible matchings none of which are stable. In the matching S_1S_2, S_3S_4, the pair S_2S_3 prefers each other. In the matching S_1S_3, S_2S_4, the pair S_1S_2 prefers each other. Finally, in the matching S_1S_4, S_2S_3, the pair S_1S_3 prefers each other.

3. *Edge-coloring.* Let G be a (not necessarily bipartite) graph with maximum degree $\Delta > 0$.

a) Show that $\chi'(G) \leq 2\Delta - 1$.

b) Suppose that G has a perfect matching M such that $G \setminus M$ is bipartite. Determine $\chi'(G)$ in terms of Δ.

Solution:
a): For fixed $\Delta > 0$ we will show by induction on $|E(G)|$ that

$$\chi'(G) \leq 2\Delta - 1$$

for every graph G with maximum degree at most Δ.
<u>Base case:</u> $|E(G)| = 0$. There is nothing to prove.

Induction step: Consider a graph G with $n \geq 1$ edges. Choose $e \in E(G)$, and let $G' = G \setminus e$. By the induction hypothesis, $\chi'(G) \leq 2\Delta - 1$ and so there exists an edge-coloring $c : E(G') \to \{1, \ldots, 2\Delta - 1\}$. Note that e shares an end with at most $2\Delta - 2$ other edges, and so there is a color available for e which is used on none of these edges. Thus we can extend the coloring by assigning this color to e.

b): As every vertex of G is incident to an edge of M, the maximum degree of $G \setminus M$ is $\Delta - 1$. Therefore we have $\chi'(G \setminus M) = \Delta - 1$ by Kőnig's theorem on edge colorings of bipartite graphs. Moreover, we can extend the coloring of $G \setminus M$ to G by using the same previously unused color on all the edges of M. Thus $\chi'(G) \leq (\Delta - 1) + 1 = \Delta$. On the other hand, we have $\chi'(G) \geq \Delta$ for every graph G. Therefore

$$\chi'(G) = \Delta.$$

4. *Counting matchings.* Let G be a graph with bipartition (A, B) such that $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_{n+1}\}$ and the vertex a_i is adjacent to vertices $b_1, b_2, \ldots, b_{i+1}$ for every $i = 1, 2, \ldots, n$. Show that there are exactly 2^n matchings in G covering A.

Solution: By induction on n.
Base case: $n = 1$. Clearly there are two ways of choosing the vertex to match a_1 to: We either choose b_1, or b_2.
Induction step ($n - 1 \to n$): By the induction hypothesis there are 2^{n-1} matchings, which match $\{a_1, a_2, \ldots, a_{n-1}\}$ to a subset of B. Note that such a macthing will leave exactly two elements of B not chosen. Therefore each of these 2^{n-1} matchings extends to exactly two matchings covering A, leaving $2^{n-1} \times 2 = 2^n$ matchings, as desired.

5. *Kőnig's theorem.* Let G be a bipartite graph with bipartition (A, B), such that $|A| = |B| = 10$, and every vertex of G has degree at least five. Show that G has a perfect matching.

Solution: By Kőnig's theorem it suffices to show that the minimum size of a vertex cover in G is eight. Let X be a vertex cover in G of minimum size. If $A \subseteq X$, or $B \subseteq X$, then clearly $|X| \geq 10$. Otherwise, there exists $v \in A - X$, which implies that all the neighbors of v lie in X, as X is a vertex cover. Therefore $|X \cap B| \geq 5$. Similarly, $|X \cap A| \geq 5$. It follows that $|X| = |X \cap A| + |X \cap B| \geq 10$, as desired.

6. *Matching markets.* Consider a matching market with with four buyers (A, B, C, D) and four sellers (X, Y, Z, W), where the valuations of the buyers are listed in the following table.

	X	Y	Z	W
A	6	4	6	6
B	6	5	7	2
C	4	1	7	5
D	3	1	6	3

Use the method seen in class to find a set of market clearing prices.

Solution: The process of generating the market clearing prices is recorded in the following table. In each row we write down the prices for the houses at the beginning of the step, the maximum potential satisfaction of all buyers and their currently preferred houses, and, finally, the set of houses for which the price is raised in this step.

X	Y	Z	W	A	B	C	D	
0	0	0	0	6(X,Z,W)	7(Z)	7(Z)	6(Z)	X,Z,W
1	0	1	1	5(X,Z,W)	6(Z)	6(Z)	5(Z)	X,Z,W
2	0	2	2	4(X,Y,Z,W)	5(Y,Z)	5(Z)	4(Z)	Z
2	0	3	2	4(X,Y,W)	5(Y)	4(Z)	3(Z)	Z
2	**0**	**4**	**2**	4(X,Y,W)	5(Y)	3(Z,W)	2(Z)	

After the last step the market clears with houses going to the following buyers:

$$A \leftarrow X, B \leftarrow Y, C \leftarrow W, D \leftarrow Z.$$