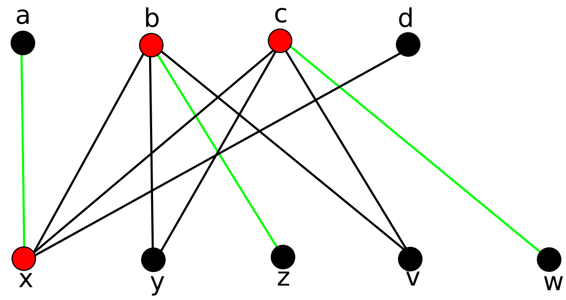


1. *Reminder:* $\tau(G)$ denotes the minimum size of a vertex cover of G, and $\nu(G)$ denotes the maximum size of a matching in G.

a) State Kőnig's theorem, relating these two parameters. [2 points]

b) Find $\nu(G)$ and $\tau(G)$ in the graph G on the figure above. [3 points]

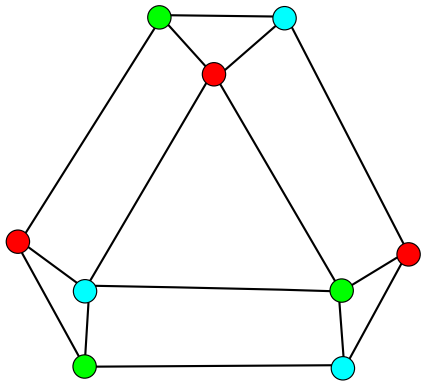
c) Let G be a bipartite graph with m edges and maximum degree d. Show that $\nu(G) \geq m/d$. [5 points]

Solution:

a): Let G be a bipartite graph. Then $\nu(G) = \tau(G)$.

b): A vertex cover of size three and a matching of size three are indicated on the figure above. Therefore $\tau(G) \leq 3 \leq \nu(G)$. It follows from Kőnig's theorem that $\tau(G) = \nu(G) = 3$.

c): Let X be a vertex cover of G. As every vertex of X is incident to at most d edges, and every edge of G is incident to a vertex of X, we have $d|X| \geq m$. Therefore $\tau(G) \geq m/d$. By Kőnig's theorem $\nu(G) \geq m/d$.



2.

a) Define the chromatic number $\chi(G)$ of a graph. [2 points]

b) Find $\chi(G)$ for the graph drawn on the figure above. [2 points]

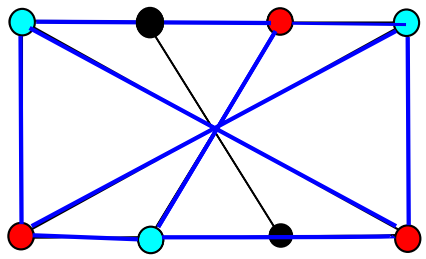
c) Let G be a graph such that $E(G)$ can be partitioned into two sets E_1 and E_2 so that $G \setminus E_1$ and $G \setminus E_2$ are both planar. Show that $\chi(G) \leq 12$.

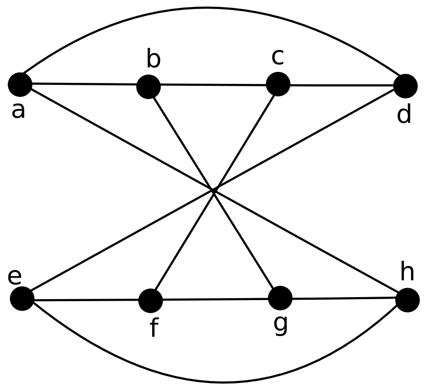
[6 points]

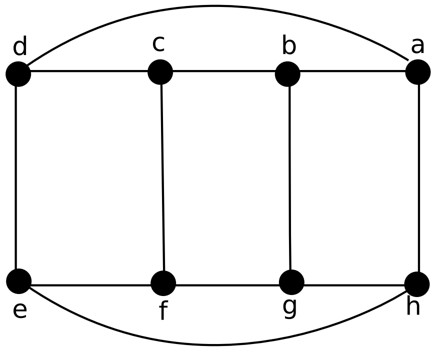
Solution:
a): The *chromatic number* $\chi(G)$ of a graph G is the minimum positive integer k such that there exists a vertex coloring of G using k colors, that is a map $c : V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for every pair of adjacent vertices u and v.

b): We have $\chi(G) \leq 3$ as the coloring using three colors is indicated on the figure above. Also, $\chi(G) \geq 3$ as G contains complete subgraph on three vertices. Therefore $\chi(G) = 3$.

c): By induction on $|V(G)|$. Base case ($|V(G)| \leq 12$) trivially holds.
Induction step. Let $n := |V(G)| \geq 13$. Using the bound on the maximum number of edges in the planar graph, we have $|E(G \setminus E_1)| \leq 3n - 6$, $|E(G \setminus E_2)| \leq 3n - 6$. Therefore, $|E(G)| = |E(G \setminus E_1)| + |E(G \setminus E_2)| \leq 6n - 12$. By the handshaking lemma, $\sum_{v \in V(G)} \deg(v) < 12n$. Therefore there exists $v \in V(G)$ such that $\deg(v) \leq 11$. By the induction hypothesis the graph $G \setminus v$ can be colored in twelve colors, and this coloring can be extended to v, implying $\chi(G) \leq 12$.







3.

a) Define a minor of a graph G. [2 points]

b) State Kuratowski's theorem. [2 points]

c) Explain whether or not each of the two graphs on the figure above is planar. [6 points]

Solution:

a): A graph H is a *minor* of a graph G if it can be obtained from G by repeatedly deleting edges and vertices and contracting edges.

b): A graph G is planar if and only if G does not contain either K_5 or $K_{3,3}$ as a minor.

c): The graph on the left is not planar, as it contains a subdivision of $K_{3,3}$ indicated on the figure as a subgraph.
The graph on the right is planar. Its planar drawing is shown below it.