MATH 340: Discrete Structures II. Winter 2016.
Due in class on Thursday, February 4th.

Assignment #1: Matchings.

1. *Stable matching algorithm.* Apply the Boy Proposal algorithm to find a stable matching given the preference lists below. Are there any other stable matchings?

$$\begin{aligned}
\mathbf{B_1} &: G_2 > G_1 > G_4 > G_5 > G_3\\
\mathbf{B_2} &: G_4 > G_2 > G_1 > G_3 > G_5\\
\mathbf{B_3} &: G_2 > G_5 > G_3 > G_4 > G_1\\
\mathbf{B_4} &: G_1 > G_4 > G_3 > G_2 > G_5\\
\mathbf{B_5} &: G_2 > G_4 > G_1 > G_5 > G_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{G_1} &: B_5 > B_1 > B_2 > B_4 > B_3\\
\mathbf{G_2} &: B_3 > B_2 > B_4 > B_1 > B_5\\
\mathbf{G_3} &: B_2 > B_3 > B_4 > B_5 > B_1\\
\mathbf{G_4} &: B_1 > B_5 > B_4 > B_3 > B_2\\
\mathbf{G_5} &: B_4 > B_2 > B_5 > B_3 > B_1
\end{aligned}$$

2. *More stable matchings.* Suppose that in a group of 100 boys and 100 girls there is a boy B, such that B is second highest on every woman's preference list. Is it possible that in every stable matching B ends up with the girl he likes least of all?

3. *Edge-coloring.* Let G be a (not necessarily bipartite) graph with maximum degree $\Delta > 0$.

a) Show that $\chi'(G) \leq 2\Delta - 1$.

b) Suppose that G has a perfect matching M such that $G \setminus M$ is bipartite. Determine $\chi'(G)$ in terms of Δ. Justify your answer.

Reminder: $G \setminus M$ is the graph obtained from G by deleting all the edges of M.

4. *Systems of distinct representatives.* Let $(S_1, S_2, \ldots, S_n)$ be a collection of subsets of $\{1, 2, \ldots, n+1\}$ such that $S_k = \{1, 2, \ldots, k+1\}$ for each $k = 1, 2, \ldots, n$. Show that there are exactly 2^n ways to chose a system of distinct representatives for $(S_1, S_2, \ldots, S_n)$.

Hint: Use induction on n.

5. *Kőnig's theorem.* Let G be a bipartite graph with bipartition (A, B), such that $|A| = |B| = 8$, and every vertex of G has degree at least four. Show that G has a perfect matching.

Hint: Show that if X is a vertex cover of G then either $|X \cap A| \geq 4$ and $|X \cap B| \geq 4$, or $A \subseteq X$, or $B \subseteq X$.

6. *Matching markets.* Consider a matching market with with four buyers (A, B, C, D) and four sellers (X, Y, Z, W), where the valuations of the buyers are listed in the following table.

	X	Y	Z	W
A	7	6	8	3
B	7	5	7	7
C	5	2	8	6
D	4	2	7	4

Use the method seen in class to find a set of market clearing prices.