MATH 240: Discrete structures I. Fall 2011

Assignment #2: Proofs. Solutions.

1. *Problems in NP.* Show that the following problems are in NP.

a) `The Knapsack problem.`

Given n items numbered 1 through n so that i-th item has weight w_i and value v_i, determine whether it is possible to select some of these items with total weight at most W and total value at least V.

Solution: This problem is in NP, because the collection of items satisfying the requirements gives a certificate for the YES answer, which can be verified in polynomial time. Given indices $i_1, i_2, \ldots, i_k$ so that

$$w_{i_1} + w_{i_2} + \ldots + w_{i_k} \leq W$$

and

$$v_{i_1} + v_{i_2} + \ldots + v_{i_k} \geq V,$$

we can verify that the above inequalities are valid in time polynomial in the input size.

b) `Quadratic Diophantine equations.`

Given three natural numbers a, b and c do there exist integers x and y so that $ax^2 + by = c$?

Solution: Suppose such x and y exist. We divide x by b with remainder: Let $x = qb + r$ for some integers q and r so that $0 \leq r < b$. Then

$$a(qb + r)^2 + by = c.$$

Expanding we obtain

$$ar^2 + b(aq^2b + 2aqr + y) = c,$$

so $x_0 = r$ and $y_0 = aq^2b + 2aqr + y$ is also a solution to the original equation. We have $|x_0| \leq b$ and therefore $|y_0| \leq ab$. Thus we can verify that $ax_0^2 + by_0 = c$ in time polynomial in the input size.

2. *Order notation.* For each of the following pairs of functions indicate whether $f = O(g)$ or $f = \Omega(g)$, or both. (All logarithms may be assumed to be natural.) In each case, briefly justify your answer.

1. $f(n) = 2n^2 + 5n, g(n) = 5n^2 + 2n.$

   Solution: $f = O(g)$ and $f = \Omega(g)$, as $f(n) \leq \frac{5}{2}g(n)$ and $g(n) \leq \frac{5}{2}$ for all n.

2. $f(n) = \log(n), g(n) = \log(n^2)$.

   Solution: $g(n) = 2\log(n) = 2f(n)$. Therefore $f = O(g)$ and $f = \Omega(g)$.

3. $f(n) = \log(n), g(n) = (\log n)^2$.

   Solution: $g(n)/f(n) = \log(n)$ and $\lim_{n\to\infty} \log(n) = \infty$. Therefore $f = O(g)$, $f \neq \Omega(g)$.

4. $f(n) = n^3 2^n, g(n) = n^2 3^n$.

   Solution: $g(n)/f(n) = n(2/3)^n \to_{n\to\infty} 0$. Therefore $f = O(g)$, $f \neq \Omega(g)$.

5. $f(n) = (logn)^n, g(n) = n^{\log n}$;

   Solution: $f(n) = e^{n \log(\log n)}, g(n) = e^{(\log n)^2}$. As $\lim_{n\to\infty} (\log n)^2/n = 0$, we have $f \neq O(g)$, $f = \Omega(g)$.

6. $f(n) = n!, g(n) = n^n$.

   Solution:
   $$\frac{f(n)}{g(n)} = \frac{1 \cdot 2 \cdot \ldots \cdot n}{n \cdot n \cdot \ldots \cdot n} \leq \frac{1}{n}.$$
   Therefore, $f = O(g)$, $f \neq \Omega(g)$.

3. *Predicate calculus.*

a) Write down in the predicate calculus the negation of the following statement:

$$\forall n \in \mathbb{N}(\exists m \in \mathbb{N}((n^2 = 3m) \vee (n^2 = 3m - 2))).$$

Solution: Using the negation rule twice we get

$$\begin{aligned}
&\neg(\forall n \in \mathbb{N}(\exists m \in \mathbb{N}((n^2 = 3m) \vee (n^2 = 3m - 2)))) \leftrightarrow \\
&\exists n \in \mathbb{N}(\neg(\exists m \in \mathbb{N}((n^2 = 3m) \vee (n^2 = 3m - 2)))) \leftrightarrow \\
&\exists n \in \mathbb{N}(\forall m \in \mathbb{N}(\neg((n^2 = 3m) \vee (n^2 = 3m - 2)))) \leftrightarrow \\
&\exists n \in \mathbb{N}(\forall m \in \mathbb{N}((n^2 \neq 3m) \wedge (n^2 \neq 3m - 2))).
\end{aligned}$$

b) Is the statement in a) true or is its negation true?

Solution: We will show that the statement in a) is true suing case analysis. The remainder of n after division by 3 is 0, 1 or 2. Therefore $n = 3k$ or $n = 3k - 2$ or $n = 3k - 1$ for some $k \in \mathbb{N}$. In the first case $n^2 = 9k^2 = 3(3k^2)$ so $m = 3k^2$ satisfies the statement. In the second case $n^2 = (3k - 2)^2 = 3(3k^2 - 4k + 2) - 2$ and $m = 3k^2 - 4k + 2$ works. Finally, if $n = 3k - 1$, we have $n^2 = 3(3k^2 - 2k + 1) - 2$.

4. *Social choice functions.* Does there exist a social choice function f satisfying the following property: For any pair of candidates α and β, if at least 60% of all the voters prefer α to β then f ranks α above β.

Solution: Such a function does not exist. Suppose for a contradiction that it does. Consider the following rankings of candidates A, B and C by 3 voters:

$$\begin{aligned} v_1 : &\quad A > B > C; \\ v_2 : &\quad B > C > A; \\ v_3 : &\quad C > A > B. \end{aligned}$$

According to the rule specified in the problem we must have $A > B$ in the ranking produced by f as 2 out of 3 voters prefer A to B. But similarly, we must have $B > C$ and $C > A$. So $A > B > C > A$, a contradiction

5. *Induction.*

a) Show that for all positive integers n

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution: Base of induction: For $n = 1$: $1^2 = 1 \cdot 2 \cdot 3/6$.

Induction step: Assuming

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

we want to show that

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.$$

By our assumption the left side of the above equation is equal to

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= (n+1)\frac{n(2n+1)+6(n+1)}{6} = (n+1)\frac{2n^2+8n+6}{6} \\ &= (n+1)\frac{(n+2)(2n+3)}{6}, \end{aligned}$$

as desired.

b) A sequence an is defined recursively by $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = 7a_{n+1} - 12a_n$ for $n \geq 0$. Show that $a_n = 4^n - 3^n$ for all non-negative integers n.

Solution: Base of induction: For $n = 0$: $a_0 = 0 = 4^0 - 3^0$. For $n = 1$: $a_1 = 1 = 4^1 - 3^1$.

Induction step: Assuming that $a_n = 4^n - 3^n$ and $a_{n+1} = 4^{n+1} - 3^{n+1}$ we have

$$\begin{aligned} a_{n+2} &= 7(4^{n+1} - 3^{n+1}) - 12(4^n - 3^n) = (7 \cdot 4 - 12)4^n - (7 \cdot 3 - 12)3^n \\ &= 16 \cdot 4^n - 93^n = 4^{n+2} - 3^{n+2}, \end{aligned}$$

as desired.

c) Show that $1 + hn \leq (1+h)^n$ for all real $h \geq -1$ and all positive integers n.

Solution: Induction on n.

Base of induction: For $n = 1$: $1 + 1 \cdot h = 1 + h \leq (1+h)^1$.

Induction step: Assuming that $1 + hn \leq (1+h)^n$ we will show that $1 + h(n+1) \leq (1+h)^{n+1}$:

$$(1+h)^{n+1} = (1+h)^n(1+h) \geq (1+hn)(1+h) = 1 + (n+1)h + nh^2 \geq 1 + (n+1)h,$$

as desired.