MATH 550: Combinatorics. Winter 2013.

Assignment # 2: Turán- and Ramsey-type problems.

Due in class on Monday, March 11th.

1. Let $(X, \mathcal{F})$ be a set system. A k-*sunflower* in $\mathcal{F}$ is a collection of distinct sets $F_1, F_2, \ldots, F_k \in \mathcal{F}$ such that for some $Z \subseteq X$ we have $F_i \cap F_j = Z$ for all $1 \leq i < j \leq k$. (The intersection of every pair of distinct sets in the sunflower is the same.)
Let $c(k, r)$ denote the maximum possible size of a set system $\mathcal{F}$ such that $|F| \leq r$ for every $F \in \mathcal{F}$, and $\mathcal{F}$ does not contain a k-sunflower. Show that

$$(k-1)^r \leq c(k,r) \leq (k-1)^r r!$$

for all $k, r \geq 1$.

2. Let $\mathcal{A} \subseteq \mathbb{N}^{(3)}$ satisfy $|\mathcal{A}| = 50$ and $|\partial\mathcal{A}| = 27$. Show that for some $Z \subseteq X \subseteq \mathbb{N}$ with $|X| = 8, |Z| = 2$, we have

$$\mathcal{A} = \{A \in \mathbb{N}^{(3)} \mid A \subset X, Z \not\subset A\}.$$

3. Let G be a graph on n vertices for some $n \geq 3$ with $|G| \geq \lfloor \frac{n^2}{4} \rfloor + 1$.

a) Show that G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

b) Show that the bound in a) is tight: For every $n \geq 3$ there exists a graph G on n vertices with $|G| = \lfloor \frac{n^2}{4} \rfloor + 1$ containing exactly $\lfloor \frac{n}{2} \rfloor$ triangles.

4. Let H be a fixed r-graph of order k. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 > 0$ with the following properties. If G is an r-graph of order $n \geq n_0$ with $|G| \geq (\pi(H)+\varepsilon)\binom{n}{r}$ then at least $\delta\binom{n}{k}$ subsets of $V(G)$ of size k induce an r-graph containing H.

5. Show that for every positive integer t there exists $\delta > 0$ such that the following holds. If G is a graph not containing K_t on n vertices and every vertex of G belongs to at least $(\frac{t-2}{t-1} - \delta)n$ edges then G is $(t-1)$-colorable.

6. Hypergraph Ramsey theorem. Show that for all positive integers r, k_1 and k_2 there exists a positive integer $n = R^{(r)}(k_1, k_2)$ so that the following holds. If elements of $[n]^{(r)}$ are colored in colors red and blue then there is a set $Z \subseteq [n]$ such that either $|Z| = k_1$ and all elements of $Z^{(r)}$ are red, or $|Z| = k_2$ and all elements of $Z^{(r)}$ are blue.
(*Hint*: Use induction on r and, for given r, induction on $k_1 + k_2$. Consider all hyperedges containing a given vertex and attempt to imitate the proof of Ramsey's theorem.)

7. Schur's theorem. Show that for every positive integer k there exists a positive integer n satisfying the following. In every coloring of $[n]$ with k colors it is possible to find a triple of (not necessarily distinct) integers x, y, z of the same color so that $x + y = z$.
(*Hint*: Use Ramsey's theorem.)