
Problem Set 2. Turán- and Ramsey-type problems.

Due in class on Thursday, March 15.

1. Bollobás. 7.4. Let $1 \leq s < r < n$ and let $F \subseteq [n]^{(r)}$ be a hypergraph such that $|A_1 \cap A_2| \leq s$ whenever $A_1, A_2 \in F$, $A_1 \neq A_2$. Show that

$$|F| \leq \frac{n(n-1)\ldots(n-s)}{r(r-1)\ldots(r-s)}.$$

2. Let $H$ be a fixed $r$-graph of order $k$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ and $n_0 > 0$ with the following properties. If $G$ is an $r$-graph of order $n \geq n_0$ with $|G| \geq (\pi(H) + \epsilon)(n^r)$ then at least $\delta(n^k)$ subsets of $V(G)$ of size $k$ induce an $r$-graph containing $H$.

3. Bollobás. 8.7. Let $K^{(3)}_4$ denote the complete $3$-graph on $4$ vertices, i.e. the $3$-graph isomorphic to $[4]^{(3)}$. Following de Caen (1983), we give an upper bound on $\pi(K^{(3)}_4)$. Let $F \subseteq [n]^{(3)}$ be a hypergraph containing no $K^{(3)}_4$ with $|F| = m$. For $x, y \in [n]$, $x \neq y$ let

$$A(x, y) := \{z \in [n] | \{x, y, z\} \in F\},$$

and let $a_{xy} := |A(x, y)|$. Note that if $\{x, y, z\} \in F$ then $A(x, y) \cap A(y, z) \cap A(z, x) = \emptyset$ and so

$$a_{xy} + a_{yz} + a_{zx} \leq 2n - 3.$$

Summing over all edges of $F$ deduce that

$$\sum_{\{x, y\} \in [n]^{(2)}} a_{xy}^2 \leq (2n - 3)m.$$

Using convexity of $x^2$ show that the left hand side is at least $(3m)^2/\binom{n}{2}$ and deduce that $m \leq \frac{2n^3}{9}\binom{n}{2}$ and $\pi(K^{(3)}_4) \leq 2/3$.

4. Hypergraph Ramsey’s theorem. Show that for all positive integers $r, k_1$ and $k_2$ there exists a positive integer $n = R^{(r)}(k_1, k_2)$ so that the following holds. If elements of $[n]^{(r)}$ are colored in colors red and blue then there is a set $Z \subseteq [n]$ such that either $|Z| = k_1$ and all elements of $Z^{(r)}$ are red, or $|Z| = k_2$ and all elements of $Z^{(r)}$ are blue.

(Hint: Use induction on $r$ and, for given $r$, induction on $k_1 + k_2$. Consider all hyperedges containing a given vertex and attempt to imitate the proof of Ramsey’s theorem.)

5. Bollobás. 20.4. Show that $R(3, 3) = 6$ and $R(3, 4) = 9$.

6. Schur’s theorem. Show that for every positive integer $k$ there exists a positive integer $n$ satisfying the following. In every coloring of $[n]$ with $k$ colors it is possible to find a triple of (not necessarily distinct) integers $x, y, z$ of the same color so that $x + y = z$.

(Hint: Use Ramsey’s theorem.)