Convex polygons in the plane.

In this note we give two proofs of a Ramsey-type classical theorem of Erdős-Szekeres on convex sets in the plane. We say that a set of points P is *in general position* if no three points in P are colinear.

Theorem 1. *For every integer n there exists an integer $g(n)$ so that any set of $g(n)$ points in the plane contains a subset of n points which form the vertex set of a convex n-gon.*

The condition that n vertices form the vertex set of a convex n-gon is equivalent to the statement that none of the points is a convex combination of the other points. By Caratheodory's theorem this is also equivalent to the statement that none of the points can be expressed as a convex combination of three other points. This observation suggest the plan for the first proof. First, we need a lemma, known as the Happy Ending theorem.

Lemma 2. *Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral.*

Proof. Consider the convex hull of the five points. It is a polygon with at least three and at most five vertices. If this polygon has at least four vertices then any four of them form a convex quadrilateral. Otherwise, the convex hull is a triangle and let A, B and C be its vertices. Denote the remaining to points by D and E. The line DE intersects exactly two sides of the triangle, without loss of generality AB and AC. Now it is easy to verify that the quadrilateral with vertices B, C, D and E is convex. □

First proof of Theorem 1. Let $g(n) := R^{(4)}(n, 5)$ be the positive integer satisfying the following. If all size 4 subsets of a set of $g(n)$ points are colored red and blue, then

- either for some 5 points all four point subsets are colored red,

- or for some n points all four point subsets are colored blue.

Such a number $g(n)$ exists by the Hypergraph Ramsey theorem established in the homework assignment.

Consider the set of $g(n)$ points in the plane and for every subset of size four color it blue, if it forms the vertices of a convex quadrilateral, and color it red, otherwise. By Lemma 2 the first of the outcomes listed above can not hold, and thus the second outcome holds for some n points. The observation preceding Lemma 2 implies that these n points form the vertex set of a convex n-gon. □

The above proof is short and appears natural. Unfortunately, the bounds on $g(n)$ it provides are superexponential and far from optimal. The second proof is slightly more technical, but gives much better bounds. Let $C = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be the set points in the plane equipped with Cartesian coordinates, and suppose that $x_1 < x_2 < \ldots < x_n$. We say that C is an *n-cup* if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \ldots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

In a cup the slopes of intervals between consecutive points increase. The set C is *an n-cap* if

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

It is not hard to verify that both cups and caps correspond to vertex sets of convex polygons. (The converse does not always holds, in a general convex polygon half of the vertex can form a cup and the other half a cap above it.)

Second proof of Theorem 1. We will prove by induction on $k + l$ that there exists a positive number $g(k, l)$ so that among any set of $g(k, l)$ points in the Cartesian plane in general position and with distinct x coordinates one can find a k-cup or an l-cap. By the observation preceding the proof this would imply the theorem.

Any 2 points form both a cap and a cup. Therefore we have $g(2, l) = g(k, 2) = 2$, establishing the base case.

For the induction step we will show that $g(k, l) = g(k-1, l) + g(k, l-1) - 1$ satisfies our claim. Let P be a collection of $N := g(k-1, l) + g(l-1, k) - 1$ points in the plane ordered according to the x coordinates. Let $A \subseteq P$ be the set of the last points of $(k-1)$-cups in P. Note that $P - A$ includes no $(k-1)$-cup. If $|P - A| \geq g(k-1, l)$ then $P - A$ contains an l-cap, as desired. Thus we assume that $|P - A| \leq g(k-1, l) - 1$ and $|A| \geq g(k, l-1)$. If A contains a k-cup the proof is finished, and so we assume that A contains an $(l-1)$-cap. The first point of this cap is the last point of some $(k-1)$-cup by the definition of A. Thus P contains a sequence of points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_{k+l-1}, y_{k+l-1})\}$ so that

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_l - y_{l-1}}{x_l - x_{l-1}},$$

and

$$\frac{y_{l+1} - y_l}{x_{l+1} - x_l} < \frac{y_{l+2} - y_{l+1}}{x_{l+2} - x_{l+1}} < \ldots < \frac{y_{k+l-1} - y_{k+l-2}}{x_{k+l-1} - x_{k+l-2}}.$$

If

$$\frac{y_l - y_{l-1}}{x_l - x_{l-1}} > \frac{y_{l+1} - y_l}{x_{l+1} - x_l}$$

then the first l vertices of this sequence form and l-cap and, otherwise, the last k vertices from a k-cup. □

Using the above proof one can show that $g(k, l) = \binom{k+l-4}{k-2} + 1$ suffices for $k, l \geq 2$. This provides an upper bound of

$$\binom{2n-4}{n-2} + 1 = \Omega\left(\frac{4^n}{\sqrt{n}}\right)$$

on the optimal value of $g(n)$ in Theorem 1. It is known that a set of 2^{n-2} points in general position does not necessarily contain a vertex set of a convex n-gon. It is conjectured that $g(n) = 2^{n-2} + 1$ satisfies Theorem 1.