Erdős-Stone theorem for graphs with chromatic number 2 and 3.

In this note we will prove a special case of the Erdős-Stone theorem, which in full generality completely determines the Turán density for all graphs. A graph H is *k-colorable* if there exists a function $c : V(H) \to [k]$ so that $c(v) \neq c(w)$ for every pair of adjacent vertices $v, w \in V(H)$. The *chromatic number* $\chi(H)$ of H is the minimum positive integer k such that H is k-colorable.

Theorem 1. *[Erdős-Stone, 1946] For every graph H with at least one edge we have*

$$\pi(H) = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

Note that the construction which is used to show that $\pi(K_t) \geq \frac{t-2}{t-1}$ implies that $\pi(H) \geq \frac{\chi(H)-2}{\chi(H)-1}$ for every graph H. Thus it suffices to establish the lower bound on the Turán density. We will prove this lower bound for graphs H with $\chi(H) = 2$ and $\chi(H) = 3$.

Let $K_{r,r}$ denote the *complete balanced bipartite graph* on $2r$ vertices, that is $V(K_{r,r}) = V_1 \cup V_2$, $|V_1| = |V_2| = r$, and $v_1 v_2 \in E(K_{r,r})$ for every $v_1 \in V_1$ and $v_2 \in V_2$. The *complete balanced* 3*-partite graph* $K_{r,r,r}$ is defined similarly with $|V(K_{r,r,r})| = 3r$, $V(K_{r,r,r}) = V_1 \cup V_2 \cup V_3$, $|V_1| = |V_2| = |V_3| = r$, and $v_i v_j \in E(K_{r,r,r})$ for all $v_i \in V_i$ and $v_j \in V_j$ for $i, j \in \{1, 2, 3\}, i \neq j$. It is easy to see that a graph H satisfies $\chi(H) \leq 2$ if and only if H is a subgraph of $K_{r,r}$ for some r, and, similarly, $\chi(H) \leq 3$ if and only if H is a subgraph of $K_{r,r,r}$ for some r. Therefore to establish Theorem 1 in the cases we are interested in it suffices to show that $\pi(K_{r,r}) = 0$ and $\pi(K_{r,r,r}) = 1/2$ for every positive integer r.The next lemma establishes the first of these identities.

For a graph G and a vertex $v \in V(G)$ let $N(v)$ denote the *neighborhood* of v, that is the set of vertices of G adjacent to v.

Lemma 2. *For every positive integer r and every $\varepsilon > 0$ there exists $n_0 > 0$ so that every graph G with $n := |V(G)| \geq n_0$ and $E(G) \geq \varepsilon n^2$ has $K_{r,r}$ as a subgraph.*

Proof. Note that it suffices to show that there exist distinct vertices $v_1, v_2, \ldots,$ $v_r \in V(G)$ such that

$$|N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| \geq r.$$

We will show that for an appropriate choice of n_0 one has

$$\sum_{\{v_1,\ldots,v_r\} \subseteq V(G)} |N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| \geq rn^r.$$

The lemma will follow by averaging. For some constant c_r depending only on r we have

$$\begin{aligned}
\sum_{\{v_1,\ldots,v_r\}\subseteq V(G)} |N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| &= \sum_{w\in V(G)} \binom{|N(w)|}{r} \\
&\geq \sum_{w\in V(G)} \left(\frac{1}{r!}\deg^r(w) - c_r \deg^{r-1}(w)\right) \\
&\geq \frac{1}{r!}\left(\sum_{w\in V(G)} \deg^r(w)\right) - c_r n^r \\
&\geq \frac{n}{r!}\left(\frac{\sum_{w\in V(G)}\deg(w)}{n}\right)^r - c_r n^r \\
&\geq \frac{n(\varepsilon n/2)^r}{r!} - c_r n^r \geq rn^r,
\end{aligned}$$

as desired. □

The next technical lemma will allow us to prove $\pi(K_{r,r,r}) = 0$ amplifying the result of Lemma 2. The proof is left as an exercise.

Lemma 3. *Let H be a fixed s-graph of order k. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 > 0$ with the following properties. If G is an s-graph of order $n \geq n_0$ with $|G| \geq (\pi(H) + \varepsilon)\binom{n}{s}$ then at least $\delta\binom{n}{k}$ subsets of $V(G)$ of size k induce an s-graph containing H.*

In this note we will only use Lemma 3 when H is a 2-graph. Let $K_{2,r}^{+}$ denote a graph on $r+2$ vertices with two of the vertices adjacent two each other and all the remaining vertices, and no other edges.

Lemma 4. $\pi(K_{2,r}^{+}) = 1/2$.

Proof. Suppose for a contradiction that $\pi(K_{2,r}^{+}) \geq 1/2 + 2\varepsilon$ for some $\varepsilon > 0$. Let δ and n_0 be chosen to satisfy Lemma 3 for $H = K_3$ and ε. Let $n' := \max\{n_0, 4r/\delta\}$ and let G be a graph of order $n \geq n'$ with $|G| \geq (1/2+\varepsilon)\binom{n}{2}$. By Lemma 3, G contains at least $\delta\binom{n}{3} \geq 3r\binom{n}{2}$ triangles, where the inequality holds by the choice of n'. It follows that some edge of G belongs to at least r triangles. Thus G contains a copy of $K_{2,r}^{+}$. It follows that $\pi(K_{2,r}^{+}) \leq 1/2 + \varepsilon$, contradicting the choice of ε. □

Lemma 5. $\pi(K_{r,r,r}) = 1/2$.

Proof. The proof follows the pattern of the proof of Lemma 4.
Suppose for a contradiction that $\pi(K_{r,r,r}) \geq 1/2 + 2\varepsilon$ for some $\varepsilon > 0$. Let δ and n_0 be chosen to satisfy Lemma 3 for $H = K_{2,r}^{+}$ and ε. Let n' be sufficiently large, which will be chosen implicitly later, depending on δ, n_0 and r, so that $n' \geq n_0$, in particular. Let G be a graph of order $n \geq n'$ with $|G| \geq (1/2+\varepsilon)\binom{n}{2}$.
For $\{v_1, v_2, \ldots, v_r\} \subseteq V(G)$, let $e_N(v_1, v_2, \ldots, v_r)$ denote the number of edges of G joining pairs of vertices in $N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)$. By Lemma 3, we have

$$\sum_{\{v_1,\ldots,v_r\}\subseteq V(G)} e_N(v_1, v_2, \ldots, v_r) \geq \frac{\delta}{r^2}\binom{n}{r+2} \geq \frac{\delta n^2}{r^4}\binom{n}{r}.$$

It follows that $e_N(v_1, v_2, \ldots, v_r) \geq \delta n^2/r^4$ for some $\{v_1, v_2, \ldots, v_r\} \subseteq V(G)$. We now apply Lemma 2 with δ/r^4 instead of ε and we assume that n' has been chosen large enough to satisfy the conclusion of this lemma. It now follows that $N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)$ contains a copy of $K_{r,r}$. Thus G

contains a copy of $K_{r,r,r}$. It follows that $\pi(K_{r,r,r}) \leq 1/2 + \varepsilon$, contradicting the choice of ε. $\square$