NAME (Print and underline last name):

STUDENT NUMBER:

SIGNATURE:

TUTORIAL FOR WHICH YOU ARE REGISTERED:

M12:35-14:25 ☐ **M14:35-16:25** ☐

W12:35-14:25 ☐ **W14:35-16:25** ☐

MATH 141
CALCULUS 2
MIDTERM
October 24, 2007, 19:00 - 21:00
VERSION 2

Instructions

- Clearly print your name and student number on this examination booklet, sign your name as indicated and fill in the time of your tutorial.
- No books, calculators or notes allowed.
- This examination booklet consists of this cover, 6 pages of questions and 2 blank pages (9 numbered pages in all).
- Answer all questions. You are expected to show all your work. All solutions are to be written on the page where the question is printed. You may continue your solutions on the facing page. If that space is exhausted you may continue on the blank pages at the end, clearly indicating any continuation on the page where the question is printed.
- Your answers may contain expressions that cannot be computed without a calculator.
- Circle your answers where confusion could arise.

GOOD LUCK!

Score Table

Problem	Points	Out of
1.		6
2.		8
3.		10
4.		12
5.		12
6.		12
Total:		60

Examiner: Georg Schmidt

V2

1. (6 marks) Identify $\sum_{i=1}^{n} \frac{1}{n} \cos\left(\frac{2\pi}{n} i\right)$ as a Riemann sum corresponding to a certain definite integral and then evaluate the limit of this quantity as $n \to \infty$.

$\int_a^b f(x)\,dx$ is defined as the limit of Riemann sums

$$\sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} \quad \text{with} \quad x_i^* \in \left[a + \frac{b-a}{n}(i-1),\ a + \frac{b-a}{n} i\right]$$

Matching quantities we see that

$$\sum_{i=1}^{n} \frac{1}{n} \cos\left(\frac{2\pi}{n} i\right) \text{ is a Riemann sum for } \int_0^1 \cos(2\pi x)\,dx$$

$$\left(\text{or for } \frac{1}{2\pi} \int_0^{2\pi} \cos x\,dx\right)$$

Hence

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cos\left(\frac{2\pi}{n} i\right) = \int_0^1 \cos(2\pi x)\,dx = 0$$

V2

2. (8 marks) Consider the function

$$f(x) = \int_0^{x^2} \frac{4-t}{1+e^{2t}} dt.$$

(a) Evaluate the derivative of $f(x)$.
(b) Show that $f'(0) = 0$ and determine the nature of the critical point 0.

a) Let $u(x) = x^2$, $g(u) = \int_0^u \frac{4-t}{1+e^{2t}} dt$

Then $f = g \circ u$ $\left(f(x) = g(u(x)) \right)$, and by the chain rule,

$$\frac{d}{dx} f(x) = \frac{d}{du} g\,(u(x)) \cdot \frac{du}{dx}$$

By the fundamental theorem of calculus, $\frac{d}{du} g(u) = \frac{4-u}{1+e^{2u}}$.

So $\boxed{\frac{d}{dx} f(x) = \frac{4-u(x)}{1+e^{2u(x)}} \cdot \frac{d}{dx}(x^2) = \frac{4-x^2}{1+e^{2x^2}} \cdot 2x}$

b) We can just substitute $x=0$ in $f'(x) = 2x \frac{4-x^2}{1+e^{2x^2}}$,

to get $\boxed{f'(0) = 2 \cdot 0 \frac{4-0}{1+e^0} = 0 \cdot \frac{4}{2} = 0}$, as wanted.

To see the nature of the critical point $x=0$, we can check that for values of x close to 0, $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$, so that f has a <u>local minimum</u> at $x=0$.

Another option would be to compute $f''(0)$ and see that it is positive, to conclude the same.

V2

3. (10 marks) Sketch the region enclosed by the curves.

$$x^2 = 8 - y \quad \text{and} \quad y - 2x = 0$$

and then

(a) evaluate the area of this region;

(b) determine the average horizontal distance (ie distance measured parallel to the x-axis) between the two curves in the region under consideration.

Intersection pts:

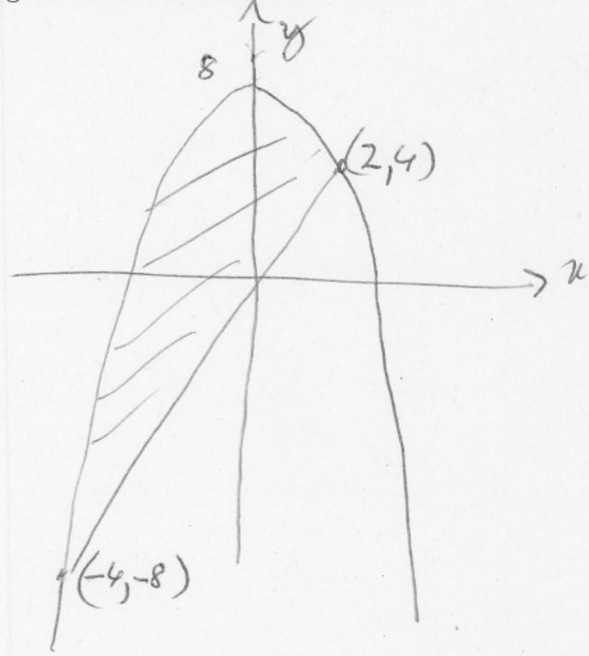
$y = 8 - x^2 = 2x$

or $x^2 + 2x - 8 = 0$

or $(x+4)(x-2) = 0$

$x = -4$ or 2

$y = -8$ or 4



(a) Area = $\int_{-4}^{2} (8 - x^2 - 2x)\, dx = \left[8x - \frac{x^3}{3} - x^2 \right]_{-4}^{2}$

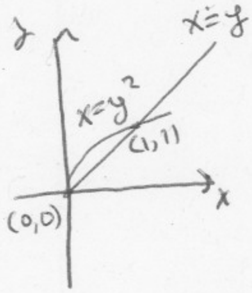
(10 marks)

$= \left(16 - \frac{8}{3} - 4\right) - \left(-32 + \frac{64}{3} - 16\right) =$

(b) (0 marks) Question was badly posed. I should have asked for the average width (measured parallel to the x axis.

Answer $\frac{36}{8-(-8)} = \frac{9}{4}$

V2



4. (12 marks) Evaluate the volumes of the three solids obtained by rotating the bounded region lying between the curves $y = x$ and $x = y^2$ about

(a) the $x-$axis; (b) the $y-$axis; (c) the line $y = 3$.

DISC

(a) $\pi \int_0^1 \left((\sqrt{x})^2 - x^2\right) dx$

$= \pi \int_0^1 x - x^2 \, dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$

$= \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \boxed{\frac{\pi}{6}}$

SHELL

(a) $2\pi \int_0^1 y(y - y^2) \, dy$

$= 2\pi \int_0^1 y^2 - y^3 \, dy$

$= 2\pi \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1$

$= 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$

(b) $\pi \int_0^1 y^2 - (y^2)^2 \, dy$

$= \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 = \boxed{\pi \left[\frac{2}{15}\right]}$

(b) $2\pi \int_0^1 x(\sqrt{x} - x) \, dx$

$= 2\pi \int_0^1 x\sqrt{x} - x^2 \, dx$

$= 2\pi \int_0^1 x^{3/2} - x^2 \, dx$

$= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^3}{3} \right]_0^1$

$= 2\pi \left[\frac{2}{5} - \frac{1}{3} \right] = \boxed{\frac{2\pi}{15}}$

(c) $\pi \int_0^1 (3 - x)^2 - (3 - \sqrt{x})^2 \, dx$

$= \pi \int_0^1 (9 - 6x + x^2) - (9 - 6\sqrt{x} + x) \, dx$

$= \pi \int_0^1 9 - 6x + x^2 - 9 + 6\sqrt{x} - x \, dx$

$= \pi \int_0^1 x^2 + 6x^{1/2} - 7x \, dx$

$= \pi \left[\frac{x^3}{3} + 6\left(\frac{2}{3}\right) x^{3/2} - \frac{7x^2}{2} \right]_0^1$

$= \pi \left[\frac{1}{3} + \frac{12}{3} - \frac{7}{2} \right]$

$= \boxed{\frac{5\pi}{6}}$

(c) $2\pi \int_0^1 (3 - y)(y - y^2) \, dy$

$= 2\pi \int_0^1 (3y - 3y^2 - y^2 + y^3) \, dy$

$= 2\pi \int_0^1 (3y - 4y^2 + y^3) \, dy$

$= 2\pi \left(3\frac{y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4} \right)_0^1$

$= 2\pi \left(\frac{3}{2} - \frac{4}{3} + \frac{1}{4} \right)$

$= 2\pi \left(\frac{5}{12} \right) = \boxed{\frac{5\pi}{6}}$

V2

5. (12 marks) Evaluate each of the following:

$$(ta) \int \frac{\ln x}{\sqrt{x}}\,dx; \qquad (b) \int \frac{\cos^3 x}{\sin^4 x}\,dx.$$

a) Integration by parts.

Let $u = \ln x$, $dv = \frac{1}{\sqrt{x}}\,dx$

Then $du = \frac{1}{x}\,dx$, $v = \int x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2x^{\frac{1}{2}} \ln x - \int 2x^{\frac{1}{2}} \cdot \frac{1}{x}\,dx$$

$$= 2x^{\frac{1}{2}} \ln x - 2\int x^{-\frac{1}{2}}\,dx$$

$$= 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} + C$$

b) $$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x}\,dx.$$

Now, let $u = \sin x$. Then $du = \cos x\,dx$

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x}\,dx = \int \frac{1-u^2}{u^4}\,du$$

$$= \int (u^{-4} - u^{-2})\,du = -\frac{1}{3}u^{-3} + \frac{1}{u} + C$$

$$= -\frac{1}{3}(\sin x)^{-3} + \frac{1}{\sin x} + C$$

$$= -\frac{1}{3}\csc^3 x + \csc x + C$$

V2

6. (12 marks) Evaluate each of the following:

$$(a) \int_0^{3\pi/2} |\sin x|\, dx; \qquad (b) \int_3^6 \frac{\sqrt{x^2-9}}{x}\, dx.$$

(a) Since $\sin x \geq 0$ for $0 \leq x \leq \pi$, $\sin x \leq 0$ for $\pi \leq x \leq 3\pi/2$,

$$|\sin x| = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ -\sin x, & \pi \leq x \leq 3\pi/2 \end{cases}.$$

and so

$$\int_0^{3\pi/2} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{3\pi/2} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx - \int_{\pi}^{3\pi/2} \sin x\, dx$$

$$= -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{3\pi/2} = -(-1) + 1 + 0 - (-1) = 3.$$

Alternatively, one can use the symmetry of the graph and evaluate the integral as

$$\int_0^{3\pi/2} |\sin x|\, dx = 3\int_0^{\pi/2} \sin x\, dx = 3, \qquad \text{say}.$$

(b) We substitute $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$. We must also change the limits of integration:

when $x = 3$, $3 = 3\sec\theta \Rightarrow \sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$,

when $x = 6$, $6 = 3\sec\theta \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$.

Therefore

$$\int_3^6 \frac{\sqrt{x^2-9}}{x}\, dx = \int_0^{\pi/3} \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta}\, 3\sec\theta\tan\theta\, d\theta = 3\int_0^{\pi/3} \tan^2\theta\, d\theta$$

$$= 3\int_0^{\pi/3} \sec^2\theta - 1\, d\theta = 3\left[\tan\theta - \theta\right]_0^{\pi/3} = 3\left(\tan\pi/3 - \pi/3\right) - 3(\tan 0 - 0)$$

$$= 3\left(\sqrt{3} - \pi/3\right) = 3\sqrt{3} - \pi$$