NAME (Print and underline last name):

STUDENT NUMBER:

SIGNATURE:

TUTORIAL FOR WHICH YOU ARE REGISTERED:

M12:35-14:25 ☐ **M14:35-16:25** ☐

W12:35-14:25 ☐ **W14:35-16:25** ☐

MATH 141
CALCULUS 2
MIDTERM
October 30, 2007, 16:00 - 18:00
ALTERNATE VERSION

Instructions

- Clearly print your name and student number on this examination booklet, sign your name as indicated and fill in the time of your tutorial.
- No books, calculators or notes allowed.
- This examination booklet consists of this cover, 5 pages of questions and 2 blank pages (8 numbered pages in all).
- Answer all questions. You are expected to show all your work. All solutions are to be written on the page where the question is printed. You may continue your solutions on the facing page. If that space is exhausted you may continue on the blank pages at the end, clearly indicating any continuation on the page where the question is printed.
- Your answers may contain expressions that cannot be computed without a calculator.
- Circle your answers where confusion could arise.

GOOD LUCK!

Score Table

Problem	Points	Out of
1.		12
2.		12
3.		12
4.		12
5.		12
Total:		60

Examiner: Georg Schmidt

1. (12 marks) The three parts of this question are independent of each other.

(a) Suppose that $\int_{-2}^{8} f(x)\,dx = 5$, $\int_{-2}^{3} f(x)\,dx = 4$ and $\int_{6}^{8} f(x)\,dx = 7$. Determine $\int_{3}^{6}(f(x)+2)\,dx$.

(b) Suppose that the derivative $f'(x)$ of $f(x)$ is continuous, that $\int_0^5 f'(x)\,dx = 10$ and that $f(5) = 12$. Determine $f(0)$?

(c) Find a continuous function $f(t)$ defined for $t \geq 0$ and a positive constant a such that

$$4 + \int_a^{x^2} f(t)\,dt = x^4$$

(a) (4 marks)

$$\int_3^6 f(x)\,dx = \int_3^{-2} + \int_{-2}^{8} + \int_8^6 = -\int_{-2}^{3} + \int_{-2}^{8} - \int_6^8 = -4+5-7 = -6$$

$$\int_3^6 (f(x)+2)\,dx = \int_3^6 f(x)\,dx + \int_3^6 2\,dx = -6+6 = 0$$

(b) (4 marks)

$$10 = \int_0^5 f'(x)\,dx = f(5) - f(0) = 12 - f(0)$$

So $f(0) = 2$

(c) (4 marks) If $x^2 = a$ get $4 + 0 = a^2$. or $a = 2$ (discard $a = -2$)

So $4 + \int_2^{x^2} f(t)\,dt = x^4$.

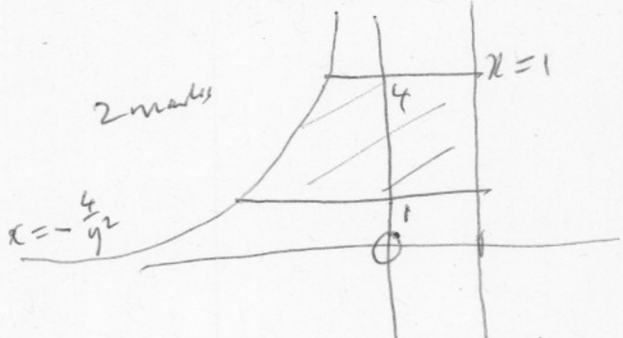
Differentiate w.r.t. x to get $f(x^2)\cdot 2x = 4x^3$ or $f(x^2) = 2x^2$

So $f(t) = 2t$ for $t \geq 0$

2. (12 marks) Sketch the region enclosed by the curves

$$x = 1, \quad x = -\frac{4}{y^2}, y = 1 \text{ and } y = 4.$$

(a) Evaluate the area of this region.
(b) Find c such that the line $y = c$ divides the region into two subregions of equal area.
(c) Find the average width (measured parallel to the x-axis) of the region.



(a) Area $= \int_1^4 \left(1 + \frac{4}{y^2}\right) dy = 3 + \left[-\frac{4}{y}\right]_1^4$
(6 marks)
$= 3 - 1 + 4 = 6$

(b) Must have $\int_1^c \left(1 + \frac{4}{y^2}\right) dy = 3.$
(2 marks)
ie $c - 1 + \left[-\frac{4}{y}\right]_1^c = 3$

$c - 1 - \frac{4}{c} + 4 = 3$

$c - \frac{4}{c} = 0$ or $c^2 = 4$ So $\boxed{c = 2}$

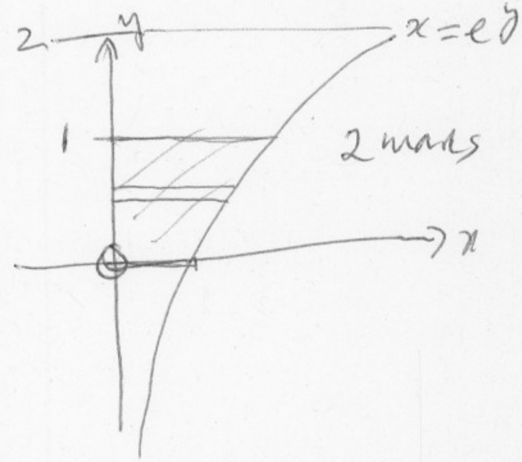
(c) Average width $= \frac{6}{4-1} = 2$

3. (12 marks) Sketch the region bounded by the four curves

$$y = \ln x, \; x = 0, \; y = 0, \; y = 1.$$

Evaluate the volumes of the two solids obtained by rotating this region about

(a) the y-axis; (b) the line $y = 2$.



2 marks

$y = \ln x \iff x = e^y$

(a) $V = \int_0^1 \pi (e^y)^2 \, dy$

5 marks

$= \pi \int_0^1 e^{2y} \, dy$

$= \pi \left[\frac{e^{2y}}{2} \right]_{y=0}^{y=1} = \frac{\pi}{2}(e^2 - 1)$

(b) $V = \int_0^1 2\pi (2-y) e^y \, dy$

5 marks

$= 2\pi \left[\int_0^1 2e^y dy - \int_0^1 y e^y dy \right] dy$

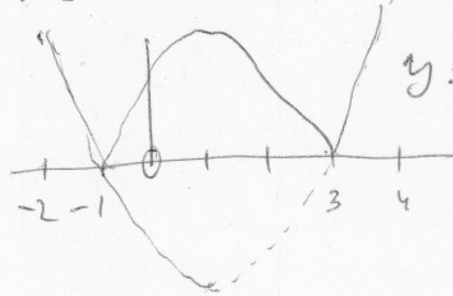
$= 2\pi \left[2 \, e^y \big|_0^1 - y e^y \big|_0^1 + \int_0^1 e^y dy \right]$

$= 2\pi \left[2e - 2 - e + e - 1 \right] = 2\pi [2e - 3]$

~~6.~~ 5 **(12 marks)** Evaluate each of the following:

$$(a) \int_{-2}^{4} |x^2 - 2x - 3|\, dx; \qquad (b) \int_{0}^{\pi/4} (\sec x + \cos x)^2\, dx.$$

(a) (5 marks) x^2-2x-3 has roots 3 and -1.



$y = |x^2-2x-3|$

Integral is equal to

$$I = \int_{-2}^{-1}(x^2-2x-3)\,dx - \int_{-1}^{3}(x^2-2x-3)\,dx + \int_{3}^{4}(x^2-2x-3)\,dx$$

Let $F(x) = \int(x^2-2x-3)\,dx = \frac{x^3}{3} - x^2 - 3x$. Then

$$I = F(-1) - F(-2) - F(3) + F(-1) + F(4) - F(3)$$

$$= F(4) - 2F(3) + 2F(-1) - F(-2) = \ldots = \frac{46}{3}$$

(b) $(\sec x + \cos x)^2 = \sec^2 x + 2 + \cos^2 x$

So $\int(\sec x + \cos x)^2 dx = \int \sec^2 x\, dx + \int 2\,dx + \int \cos^2 x\, dx$

$$= \tan x + 2x + \frac{1}{2}\int(1+\cos 2x)\,dx$$

$$= \tan x + \frac{5}{2}x + \sin 2x$$

$$\int_0^{\pi/4}(\sec x + \cos x)^2 dx = \tan\frac{\pi}{4} + \frac{5}{8}\pi + \frac{1}{4}\sin\frac{\pi}{2}$$

$$= 1 + \frac{5}{8}\pi + \frac{1}{4} = \frac{5}{4} + \frac{5\pi}{8}$$

5. **(12 marks)** Evaluate each of the following:

$$(a) \int \cos\sqrt{\theta}\, d\theta; \qquad (b) \int \frac{x}{\sqrt{8-2x-x^2}}\, dx.$$

(a) $u = \sqrt{\theta}$ $\quad du = \frac{1}{2\sqrt{\theta}}\, d\theta$ $\quad$ so $d\theta = 2u\, du$

(6 marks)

$$\int \cos\sqrt{\theta}\, d\theta = 2\int u \cos u\, du = 2u\sin u - 2\int \sin u\, du$$

$$= 2u\sin u + 2\cos u + C$$

$$= 2\sqrt{\theta}\sin\sqrt{\theta} + 2\cos\sqrt{\theta} + C$$

(b) $8 - 2x - x^2 = 9 - (x+1)^2$

(6 marks)

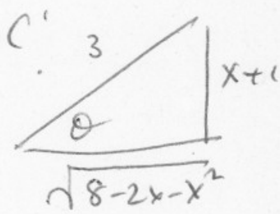
Set $x+1 = 3\sin\theta$ so $dx = 3\cos\theta\, d\theta$

and integral becomes

$$\int \frac{3\sin\theta - 1}{\cancel{3\cos\theta}}\, \cancel{3\cos\theta}\, d\theta$$

$$= \int (3\sin\theta - 1)\, d\theta = -3\cos\theta - \theta + C$$

$$= -3\,\frac{\sqrt{8-2x-x^2}}{3} - \sin^{-1}\frac{x+1}{3} + C'$$



$$= -\sqrt{8-2x-x^2} - \sin^{-1}\frac{x+1}{3} + C$$