NAME (underline family name):

STUDENT NUMBER: SIGNATURE:

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 141

CALCULUS 2

Examiner: G. Schmidt **Date:** Tuesday, December 11, 2007
Associate Examiner: W. Brown **Time:** 9:00 AM - 12:00 AM

Instructions

1. Write your name and student number on this examination script.
2. No books, calculators or notes allowed.
3. This examination booklet consists of this cover, 10 pages of questions and 2 blank pages (12 numbered pages in all). Please take a couple of minutes in the beginning of the examination to scan the problems. (Please inform the invigilator if the booklet is defective.)
4. Answer all questions. You are expected to show all your work. All solutions are to be written on the page where the question is printed. You may continue your solutions on the facing page. If that space is exhausted you may continue on the blank pages at the end, clearly indicating any continuation on the page where the question is printed.
5. Your answers may contain expressions that cannot be computed without a calculator.
6. Circle your answers where confusion could arise.

GOOD LUCK!

Score Table

Problem	Points	Out of
1.		8
2.		12
3.		10
4.		10
5.		10
6.		10
7.		10
8.		10
9.		12
10.		8
Total:		100

1. (total of 8 marks)

(a) (4 marks) Identify $R_n = \sum_{i=1}^{n} \frac{i}{n^2} e^{-2i/n}$ as a Riemann sum corresponding to a certain definite integral.

(b) (4 marks) Evaluate $\lim_{n\to\infty} R_n$.

(a) $\int_a^b f(x)\,dx$ is defined as the limit of Riemann sums

$$\sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} \quad \text{with } x_i^* \in \left[a + \frac{b-a}{n}(i-1),\ a + \frac{b-a}{n} i\right]$$

Matching quantities we see that

$$\sum_{i=1}^{n} \frac{i}{n^2} e^{-2i/n} \text{ is a Riemann sum for } \boxed{\int_0^1 x e^{-2x}\,dx.}$$

(b)
$$\lim_{n\to\infty} R_n = \int_0^1 x e^{-2x}\,dx$$
$$= -\frac{1}{2} e^{-2x} \cdot x \Big|_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\,dx$$
$$= -\frac{1}{2} e^{-2} - \left[\frac{1}{4} e^{-2x}\right]_{x=0}^{x=1}$$
$$= -\frac{1}{2} e^{-2} - \frac{1}{4} e^{-2} + \frac{1}{4} = \boxed{\frac{1}{4} - \frac{3}{4} e^{-2}}$$

2. (total of 12 marks)
(a) (6 marks) Find a continuous function $f(t)$ defined for $t \geq 0$ and a positive constant a such that

$$1+\int_a^{x^2} e^t f(t)\,dt = \ln(1+x^2)$$

(Hint: to find $f(t)$ differentiate both sides of the equation with respect to x.)
(b) (6 marks) Evaluate $\int_{-3}^{4} |x^2 - 2x - 3|dx$.

(a) Differentiating w.r.t. x we get

$$e^{x^2} f(x^2) \cdot 2x = \frac{2x}{1+x^2}$$

So $f(x^2) = \dfrac{e^{-x^2}}{1+x^2}$ and $\boxed{f(t) = \dfrac{e^{-t}}{1+t}}$

Substituting $x=\sqrt{a}$ into the equation (so $x^2=a$)
get $1+0 = \ln(1+a)$

i.e. $\ln(1+a) = 1$

So $1+a = e$ or $\boxed{a = e-1}$

(b) $x^2-2x-3 = (x-3)(x+1)$
So x^2-2x-3 is positive for $x>3$ and $x<-1$
and negative between -1 and 3.

Hence $I = \int_{-3}^{4} |x^2-2x-3|\,dx = \int_{-3}^{-1}(x^2-2x-3)\,dx - \int_{-1}^{3}(x^2-2x-3)\,dx$
$+ \int_{3}^{4}(x^2-2x-3)\,dx.$

The antiderivative of x^2-2x-3 is
$F(x) = \frac{1}{3}x^3 - x^2 - 3x$

Hence
$$I = (F(-1)-F(-3)) - (F(3)-F(-1)) + (F(4)-F(3))$$
$$= 2F(-1) - F(-3) - 2F(3) + F(4)$$
$$= 2\left(-\tfrac{1}{3}-1+3\right) - (-9-9+9) - 2(9-9-9) + \left(\tfrac{64}{3}-16-12\right)$$
$$= \frac{10}{3} + 9 + 18 - \frac{20}{3} = \frac{71}{3}$$

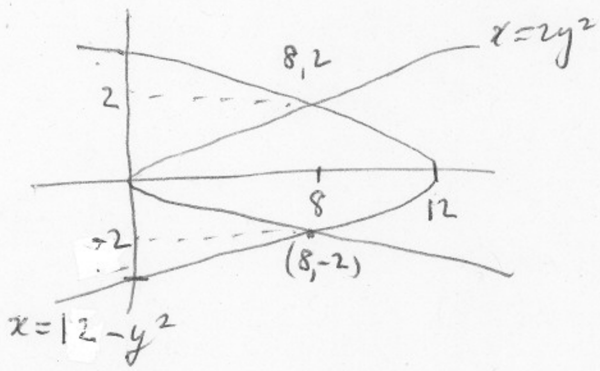
3. (total of 10 marks)

(a) (7 marks) Find the area of the region bounded by the two curves

$$x + y^2 = 12 \text{ and } x - 2y^2 = 0$$

(b) (3 marks) Find the average vertical height (measured parallel to the y-axis) of the region considered in (a).

(a) The curves are parabolas
$x = 12 - y^2$, $x = 2y^2$
which intersect when
$12 - y^2 = 2y^2$ i.e.
for $y = \pm 2$, $x = 8$



$$\text{Area} = \int_{-2}^{2} [12 - y^2 - 2y^2]\, dy = \int_{-2}^{2} (12 - 3y^2)\, dy$$

$$= \left[12y - y^3\right]_{-2}^{2} = 32$$

(b) The average vertical height is $\dfrac{\text{Area}}{12-0} = \dfrac{32}{12} = \dfrac{8}{3}$

4. (total of 10 marks, 5 for each integral) Evaluate each of the following definite integrals

$$(a)\ \int_2^4 \frac{x^2}{x^2-x}dx; \qquad (b)\ \int_1^2 \frac{x^2}{\sqrt{3+2x-x^2}}dx.$$

(a) $\frac{x^2}{x^2-x} = \frac{x^2-x}{x^2-x} + \frac{x}{x^2-x} = 1 + \frac{1}{x-1}$

So $\int \frac{x^2}{x^2-x}dx = x + \ln|x-1| \ (+C)$

So $\int_2^4 \frac{x^2}{x^2-x}dx = (4+\ln 3) - (2+\ln 1) = 2 + \ln 3$

(b) $I = \int_1^2 \frac{x^2}{\sqrt{4-(x-1)^2}}dx$

Set $x-1 = 2\sin\theta$ so $dx = 2\cos\theta\, d\theta$, $\sqrt{4-(x-1)^2} = 2\cos\theta$

Also $x=1$ corresponds to $\theta=0$ while $x=2$ corresponds to $\theta=\pi/6$

Hence

$$I = \int_0^{\pi/6} \frac{(1+2\sin\theta)^2}{\cancel{2\cos\theta}} \cancel{2\cos\theta}\, d\theta = \int_0^{\pi/6} (1+4\sin\theta+4\sin^2\theta)\, d\theta$$

$$= \int_0^{\pi/6} (1+4\sin\theta+2(1-\cos 2\theta))\, d\theta$$

$$= \int_0^{\pi/6} (3+4\sin\theta-2\cos 2\theta)\, d\theta$$

$$= \Big[3\theta - 4\cos\theta - \sin 2\theta\Big]_0^{\pi/6}$$

$$= 3\pi/6 - 4\left(\tfrac{\sqrt{3}}{2}-1\right) - \left(\tfrac{\sqrt{3}}{2}-0\right)$$

$$= \frac{\pi}{2} - \frac{5\sqrt{3}}{2} + 4$$

5. (total of 10 marks, 5 for each integral) Evaluate each of the following indefinite integrals

$$(a)\ \int x(\ln x)^2 dx; \qquad (b)\ \int \frac{(x^2-4)^{3/2}}{x} dx.$$

(a) Integrate by parts (twice):

$$\int x(\ln x)^2 dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}\int x^2 \cdot 2\ln x \cdot \frac{1}{x}\, dx$$

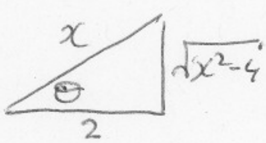
$$= \frac{1}{2}x^2(\ln x)^2 - \int x \ln x \, dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{2}\int x^{\cancel{2}} \frac{1}{\cancel{x}}\, dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$$

(b) With $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta$

$$\int \frac{(x^2-4)^{3/2}}{x} dx = \int \frac{(4\tan^2\theta)^{3/2}}{2\sec\theta} 2\sec\theta\tan\theta \, d\theta$$



$$= 8\int \tan^4\theta \, d\theta$$

$$= 8\int \tan^2\theta(\sec^2\theta - 1)\, d\theta$$

$$= 8\int [\tan^2\theta \sec^2\theta - \tan^2\theta]\, d\theta$$

$$= 8\int [\tan^2\theta \sec^2\theta - \sec^2\theta + 1]\, d\theta$$

$$= 8\left[\frac{1}{3}\tan^3\theta - \tan\theta + \theta\right] + C$$

$$= 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)^3 - \frac{\sqrt{x^2-4}}{2} + \operatorname{Arcsec}\frac{x}{2}\right] + C$$

$$\left(\operatorname{Arcsec}\frac{x}{2} = \operatorname{Arccos}\frac{2}{x}\right)$$

6. (total of 10 marks)

(a) (6 marks) Evaluate $\int \frac{3x-7}{(x+1)(x^2+4)}\,dx$.

(b) (4 marks) Check whether the following integral is convergent or divergent and, if possible, find its value (finite or infinite) : $\int_2^\infty \frac{3x-7}{(x+1)(x^2+4)}\,dx$.

(a) $$\frac{3x-7}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{Ax^2+4A+Bx^2+Bx+Cx+C}{(x+1)(x^2+4)} = \frac{(A+B)x^2+(B+C)x+4A+C}{(x+1)(x^2+4)}$$

Hence $A+B=0$, $B+C=3$, $4A+C=-7$.
Multiplying through by $x+1$, setting $x=-1$ get $A=-2$. Hence $B=2$ and $C=1$.

$$\frac{3x-7}{(x+1)(x^2+4)} = -\frac{2}{x+1} + \frac{2x+1}{x^2+4} = -\frac{2}{x+1} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}$$

So $$\int \frac{3x-7}{(x+1)(x^2+4)}\,dx = -2\ln|x+1| + \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

(b) $$\int_2^R \frac{3x-7}{(x+1)(x^2+4)}\,dx = -2\ln(R+1) + \ln(R^2+4) + \frac{1}{2}\tan^{-1}\frac{R}{2}$$

$$+2\ln 3 - \ln 8 - \frac{1}{2}\tan^{-1}1$$

$$-2\ln(R+1) + \ln(R^2+4) + \frac{1}{2}\tan^{-1}\frac{R}{2} = \ln\frac{R^2+4}{(R+1)^2} + \frac{1}{2}\tan^{-1}\frac{R}{2}$$

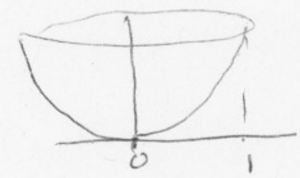
$$\to \ln 1 + 0 \text{ as } R\to\infty$$

So $\int_2^\infty \frac{3x-7}{(x+1)(x^2+4)}\,dx$ is convergent to

$$2\ln 3 - \ln 8 - \frac{\pi}{8} \quad (\text{since } \tan^{-1}1 = \frac{\pi}{4})$$

$$= \ln\frac{9}{8} - \frac{\pi}{8}$$

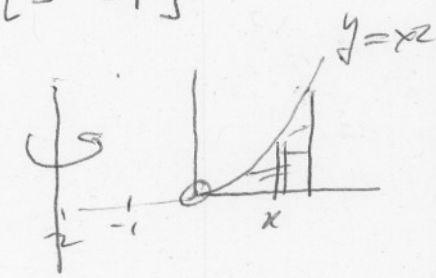
7. (total of 10 marks)
(a) (5 marks) Find the area of the surface obtained by rotating the arc of the parabola $y = x^2$ lying between $x = 0$ and $x = 1$ about the y-axis.
(b) (5 marks) Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x = 0$, $x = 1$ and the x axis about $x = -2$



(a) $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+4x^2}\, dx$

$$\text{Area} = 2\pi \int_0^1 x\, ds = 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx$$

$$= 2\pi \frac{2}{3}\cdot\frac{1}{8}\left(1+4x^2\right)^{3/2}\Big|_{x=0}^{x=1} = \frac{\pi}{6}\left[5^{3/2}-1\right]$$



(b) By cylindrical shells

$$\text{Vol} = 2\pi \int_0^1 (2+x)\, x^2\, dx$$

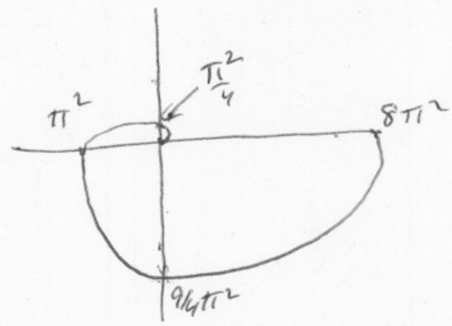
$$= 2\pi \int_0^1 (2x^2+x^3)\, dx = \frac{4\pi}{3} x^3\Big|_{x=0}^{x=1} + \frac{\pi}{2} x^4\Big|_{x=0}^{x=1}$$

$$= \frac{4\pi}{3} + \frac{\pi}{2} = \frac{11\pi}{6}$$

One can also set up the integrals w.r.t. y ending up with the same answers

8. (total of 10 marks) Consider the polar curve $r = \theta^2$.
(a) (2 marks) Sketch the curve for the range $0 \leq \theta \leq 2\pi$.
(b) (4 marks) Find the arc length of the curve from $\theta = 0$ to $\theta = \pi/2$
(c) (4 marks) Find the area enclosed by the x-axis, the y-axis and the section of the curve from $\theta = 0$ to $\theta = \pi/2$.

(a)



(b) $$L = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{\frac{\pi}{2}} \theta\sqrt{\theta^2+4}\, d\theta$$

$$= \frac{2}{3}\cdot\frac{1}{2}(4+\theta^2)^{3/2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}\left[\left(4+\frac{\pi^2}{4}\right)^{3/2} - 8\right]$$

(c) $$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \theta^4\, d\theta = \frac{1}{10}\theta^5\Big|_0^{\pi/2}$$

$$= \frac{1}{10}\left(\frac{\pi}{2}\right)^5 = \frac{\pi^5}{320}$$

9. (total of 12 marks, 4 for each series) Determine whether each of the following series is convergent or divergent, specifying the tests you use and verifying the conditions which let you apply the test:

$$(a)\ \sum_{k=1}^{\infty} \frac{2^k(k+1)}{k!}; \qquad (b)\ \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^{3/2}}; \qquad (c)\ \sum_{k=1}^{\infty} \frac{2+\sin k}{(k+1)^2}.$$

(a) Apply ratio test $\dfrac{a_{k+1}}{a_k} = \dfrac{2^{k+1}(k+2)}{(k+1)!} \Big/ \dfrac{2^k(k+1)}{k!}$

$$= \frac{2}{k+1} \cdot \frac{k+2}{k+1} \to 0$$

So series converges.

(b) Use integral test $a_k = f(k)$ with $f(x) = \dfrac{1}{x(\ln x)^{3/2}}$

For $x \geq 2$ this is decreasing, positive and continuous.

$$\int \frac{1}{x(\ln x)^{3/2}}\,dx = -2\frac{1}{(\ln x)^{1/2}} \quad (u = \ln x \text{ gives this})$$

So $\displaystyle\int_2^R \frac{1}{x(\ln x)^{3/2}} = -2\frac{1}{(\ln R)^{1/2}} + \frac{2}{(\ln 2)^{1/2}}$

$$\to \frac{2}{(\ln 2)^{1/2}} \text{ as } R \to \infty.$$

Since $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}\,dx$ is convergent, so is the given series

(c) $0 \leq \dfrac{2+\sin k}{(k+1)^2} \leq \dfrac{3}{k^2}$. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2}$ is a convergent p-series

So the given series converges by the comparison test.

10. (total of 8 marks, 4 for each series) Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent, specifying the tests you use and verifying the conditions which let you apply the test:

$$(a)\ \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1-\ln k}; \qquad (b)\ \sum_{k=1}^{\infty} (-1)^k \frac{(k+2)^2}{k^2} \cos(\pi/k).$$

(a) $\frac{1}{2k+1-\ln k}$ is positive and decreasing

(since $\frac{d}{dx}(2x+1-\ln x) = 2-\frac{1}{x} > 0$ for $x \geq 1$ and since it is positive for $k=1$)

Also $\lim_{k\to\infty} \frac{1}{2k+1-\ln k} = \lim_{k\to\infty} \frac{1/k}{2+1/k-\frac{\ln k}{k}} = 0$.

The series converges by the AST.

Check for absolute convergence.

$\sum_{k=1}^{\infty} \frac{1}{2k+1-\ln k}$ is divergent by limit comparison with $\sum_{k=1}^{\infty} \frac{1}{k}$ since

$|a_k| / \frac{1}{k} = \frac{k}{2k+1-\ln k} = \frac{1}{2+\frac{1}{k}-\frac{\ln k}{k}} \to \frac{1}{2}$

and since $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series.

The series is therefore conditionally convergent.

(b) $|a_k| = \frac{(k+2)^2}{k^2} \cos\frac{\pi}{k} = \left(1+\frac{2}{k}\right)^2 \cos\frac{\pi}{k} \to 1$ as $k\to\infty$

So $a_k \not\to 0$ and series is divergent by "divergence test"