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THE CONSERVATION LAW $\partial_y u+\partial_x\sqrt{1-u^2}=0$ AND DEFORMATIONS OF FIBRE-REINFORCED MATERIALS*

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Abstract. The conservation law $\partial_y u + \partial_x \sqrt{1-u^2} = 0$ is found to govern planar deformations of incompressible materials containing a continuous linear distribution of inextensible fibres. Kinematically feasible deformations are discussed, with emphasis on admissibility and the resolution of nonuniqueness. Many of the aspects of hyperbolic conservation laws have direct consequences in the kinematics of these materials, thus providing an illustrative guide to the theory. Alternatively, the study of this conservation law is geometrically motivated by questions on the structure of the set of points above a continuous function curve whose minimum distance to the curve is achieved in several places.

Key words. conservation law, fibre-reinforced materials, characteristics

AMS subject classifications. 35L65, 35L67, 73B40, 73K20

1. Introduction to the problem.

In this article a particular hyperbolic conservation law in a rather unusual setting is studied. The equation governs deformations of certain idealized composite materials: planar deformations of an incompressible matrix reinforced with inextensible fibres (cf. Pipkin and Rodgers [9], Spencer [10]). A hyperbolic conservation law is a first-order, quasi-linear equation of the form

$$\partial_y u + \partial_x[f(u)] = 0, \tag{1.1}$$

where f is some smooth and usually nonlinear function. The mathematical theory behind such equations began in the fifties with a paper by Hopf (cf. [5]) on the special case $f(u) = \frac{1}{2}u^2$ and was further developed for (1.1) by Lax and Oleinik (cf. [6], [7]). The fact that (1.1) has always been regarded as an evolution equation (y being a time variable) has introduced concepts such as entropy which have been guiding forces in the analysis of solutions to initial value problems. We will be interested here in a purely spatial setting, where solution values are specified either explicitly or implicitly on a function curve. In addition to having an application to kinematics of composites, the study of our equation provides a vivid illustration of many facets of hyperbolic conservation laws; many aspects of the theory have direct physical significance in the fibre configurations of these materials.

Alternatively, the study of our equation plays a central role in describing the following set. Given a continuous function curve $y = \Theta(x)$, what is the structure of the set of points above the curve whose minimum distance to the curve is attained at more than one (or two) point(s)? We show that regardless of the differentiability of Θ, this set consists of a countable number of Lipschitz curves which are smooth[1] almost everywhere (a.e). The set of points which have more than two distance minimizers is at most countable and these points are either points of intersection or initial points of these Lipschitz curves. This increased regularity is a consequence of the genuine nonlinearity (cf. [6]) of the problem, which is made transparent by consideration of the conservation law. Finally, the conservation law studied in this purely

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[1] Unless additional information is given, smooth means continuously differentiable.

spatial situation bridges two fundamental aspects of the theory of scalar conservation laws: minimization via the Hamilton–Jacobi theory (the Lax characterization) and generalized characteristics.

The physical problem is the following. Consider an incompressible material (for example, rubber) which contains a continuous distribution of strong fibres (for example, metal). The purely kinematic study of such materials began in the early seventies with work of Pipkin and Rodgers [9], [10] who proposed a theory irrespective of the nature of the materials in question. Working with the assumptions that the composite was incompressible and the fibres were inextensible and continuously distributed, they considered planar deformations of a block with initially parallel linear fibres. The problem was to determine kinematically feasible deformations, hence fibre configurations, of the entire block given the deformation of a particular boundary fibre. The following conclusions were made. In general, such a boundary displacement problem has many possible fibre configurations, some of which may involve discontinuities in the fibre direction. Distinguishing among these configurations is not possible on kinematic grounds alone. Because the assumptions of incompressibility and inextensibility result in an arbitrary pressure and tension component in the entire stress function, given reasonable assumptions on the stress response (i.e., the particular material type), any kinematically admissible deformation can be achieved, i.e., solve the equations of equilibrium. Thus kinematically admissible solutions are statically admissible.

We introduce the basic notation and derive the conservation law at work. Let $\mathbf{X} = (X_1, X_2, X_3)$ denote the position of a material point in the reference configuration and $\mathbf{x}(\mathbf{X}) = (x_1(\mathbf{X}), x_2(\mathbf{X}), x_3(\mathbf{X}))$ the position in the deformed configuration where the deformation $\mathbf{x}(\mathbf{X})$ is a Lipschitz homeomorphism. Let $F = \nabla \mathbf{x}$ be the deformation gradient and assume we are dealing with a planar deformation, i.e., $x_3(\mathbf{X}) = X_3$ and x_1, x_2 are functions of X_1 and X_2 only. The composite material is said to be incompressible if the set of admissible deformations is contained in $\{\, \mathbf{x}(\mathbf{X}) \,:\, \det \nabla \mathbf{x} = 1 \text{ for a.e. } \mathbf{X} \,\}$. For simplicity of notation let $X_1 = X, X_2 = Y, x_1 = x$, and $x_2 = y$. We assume that through each point $\mathbf{X}$ passes a fibre with unit tangent vector $\mathbf{A}(\mathbf{X})$. Thus the reference fibres are trajectories of the vector field $\mathbf{A}$. We take $A_3 = 0$. The vector field

$$\mathbf{a}(\mathbf{X}) = F \cdot \mathbf{A}(\mathbf{X})$$

will determine the position of the fibres in the deformed body. The above gives a referential (Lagrangian) description of the deformed fibre tangent vectors. We use $\mathbf{x}(\mathbf{X})$ to write $\mathbf{a}$ as a function of $\mathbf{x}$, hence x and y, and from now on refer to $\mathbf{a}$ as this function, which is our basis for describing the deformed state of the composite. Our dependent variables are thus a_1 and a_2, the first and second components of $\mathbf{a}$, and $a_2(x, y)$, for example, represents the vertical component of the deformed fibre which passes through the point (x, y). Regardless of whether the deformation is planar or not, the composite being incompressible implies that the divergence of the fibre tangent vector field is conserved in the following sense.

PROPOSITION 1.1. *If the deformation* $\mathbf{x}$ *preserves volume, i.e.,* $\det \nabla \mathbf{x} = 1$ *a.e., then*

$$\operatorname{div} \mathbf{a} = \mathrm{DIV}\ \mathbf{A}, \tag{1.2}$$

where the equality is interpreted in the sense of distributions. Here, div *is with respect to the spatial variables* x, y, *and* DIV *is with respect to the material variables* X, Y.

Proof. This proposition is attributed to Pipkin and Rodgers in the literature (cf. [10]) and a proof for smooth vector fields can be found in either [9] or [10]. We will be concerned with vector fields which may not even be piecewise smooth, and hence we show (1.2) in the sense of distributions. Adopting the usual summation convention, let $\phi \in C_c^\infty$ be any test function. Then

$$\langle \text{div } \mathbf{a}, \phi \rangle = \int a_i \frac{\partial \phi}{\partial x_i} \, d\mathbf{x} = \int \frac{\partial x_i}{\partial X_j} A_j \frac{\partial \phi}{\partial x_i} \, d\mathbf{x}.$$

Using the incompressibility condition (determinant of deformation gradient is one a.e.),

$$\langle \text{div } \mathbf{a}, \phi \rangle = \int \frac{\partial x_i}{\partial X_j} A_j \frac{\partial \phi}{\partial x_i} \, d\mathbf{X} = \int A_j \frac{\partial \phi}{\partial X_j} \, d\mathbf{X} = \langle \text{DIV } \mathbf{A}, \phi \rangle. \qquad \square$$

The inextensibility of the fibres is equivalent to a being a unit vector, so $(a_1)^2 + (a_2)^2 = 1$. Under the assumption that a_1 does not change sign, say positive, and hence the fibres all look like graphs of functions of x, we have

$$a_1 = \sqrt{1 - (a_2)^2}. \tag{1.3}$$

Assuming the initial configuration of the fibres consists of parallel lines, i.e., $\mathbf{A} \equiv (1, 0, 0)$, we combine the linear equation (1.2) with (1.3) to obtain the nonlinear equation

$$\partial_y a_2 + \partial_x \sqrt{1 - (a_2)^2} = 0.$$

2. Solutions: Breakdown of continuous fibre directions and existence of weak solutions. Let us agree to refer to the dependent variable a_2 as u. Thus we will study

$$\partial_y u + \partial_x \sqrt{1 - u^2} = 0. \tag{2.1}$$

Consider the region in the reference domain $\{(X, Y) \,:\, -\infty < X < \infty,\ 0 \le Y \le \infty\}$. Suppose it is known that $Y = 0$ is deformed into the curve $y = \Theta(x)$. What deformations are possible, or, alternatively, what kinematically feasible deformed fibre configurations are possible? For $\Theta(x)$ continuous, consider the set

$$B := \{(x, y) \,:\, y \ge \Theta(x)\},$$

and let B^o denote its interior $\{(x, y) \,:\, y > \Theta(x)\}$. To begin, we assume Θ is differentiable and look for a solution to (2.1) on B such that

$$u(x, \Theta(x)) = \frac{\Theta'(x)}{\sqrt{1 + [\Theta'(x)]^2}}.$$

We refer to this problem as the upward problem, i.e., determining fibre configurations from a specified bottom fibre. Once $u(x, y)$ is found, the deformed fibre configuration is obtained by solving the system

$$\frac{dx}{dt} = \sqrt{1 - u^2}, \qquad \frac{dy}{dt} = u, \tag{2.2}$$

whose trajectories correspond to the deformed fibres, noting that regardless of the regularity of u, the solution $(x(t), y(t))$ must be the continuous image of a line with $\mathbf{X}$ constant. This new configuration provides an illustration of the kinematically feasible deformation $\mathbf{x}$. Let us note that a great deal of information about (2.1) is known. This knowledge dates back to Lax [6] and Oleinik [7] who, following a paper of Hopf [5] pertaining to a quadratic f, laid the foundation for hyperbolic conservation laws. One might hastily conclude that everything is known about (2.1). However, there are some subtle but important differences; namely, y is not interpreted as a time variable and initial conditions are prescribed on a function curve rather than on a variable axis. Moreover, there is an intimate relationship between the data and where they are prescribed. Uniqueness is a central problem in the study of (1.1), and analytical methods alone do not suffice to alleviate the nonuniqueness present in global solutions. One must go to the physics surrounding the conservation law to weed out the physically undesirable solutions. For the evolution equations this centres around the idea of entropy and has led to irreversibility of y in the initial value problem.

As with any scalar conservation law, the solution to (2.1) is constant along classical characteristic lines which in the present situation are normal lines to the boundary Θ fibre, indeed, to every fibre. Recall that classical characteristics are solutions to the ordinary differential equation

$$\frac{dx}{dy} = f'(u) = \frac{-u}{\sqrt{1-u^2}}.$$

In fact, (2.1) is simply a statement that the normal derivative of u along any fibre is 0. Hence given a specified Θ curve with simple structure, it is easy to sketch a feasible deformed fibre configuration and therefore a solution u by constructing "parallel" curves. One immediately realizes that the normal distance between any two initially parallel fibres is preserved everywhere, a result obtained by Pipkin and Rodgers using geometry. They consider the Frenet–Serret equations of the fibres together with (1.2) to obtain equality between the curvature of the normal curves (curves normal to the fibres) in the deformed and initial configurations. If a crimp line occurs (a curve across which fibre directions are discontinuous or alternatively where characteristics collide), the Rankine–Hugoniot jump conditions (cf. [6]) for (2.1) imply that the fibres on each side of the crimp line make equal angles with the crimp line. Precisely, the Rankine–Hugoniot jump conditions imply that the slope, with respect to y, of the crimp line is related to the jump in fibre directions by

$$s = \frac{\sqrt{1-(u_+)^2} - \sqrt{1-(u_-)^2}}{u_+ - u_-},$$

where $u_\pm$ are limits from the left and right, respectively. Examples of solutions to (2.1), i.e., kinematically feasible deformations, for various bottom fibre specifications are given in Figures 1 through 7; Figure 1 shows the undeformed block. The specified boundary $y = \Theta(x)$ is denoted in bold. If Θ is concave down and smooth, the problem has a unique solution with no crimp line (see Figure 2). However, if it is not smooth, one can have many solutions, some with crimp lines and others with smooth fibres (see Figures 3, 4). If Θ is concave up, crimp lines appear in any feasible deformation (see Figures 5, 6, and 7). One might hastily conclude that in such a case we have a unique solution, but Figure 6 shows that a continuum of solutions, parametrized by the angle between adjacent crimp lines, can be constructed for the same boundary

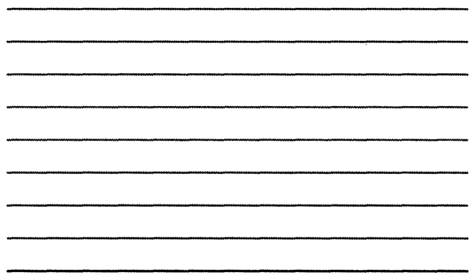

FIG. 1.

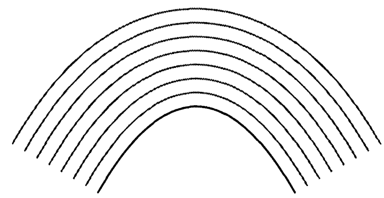

FIG. 2.

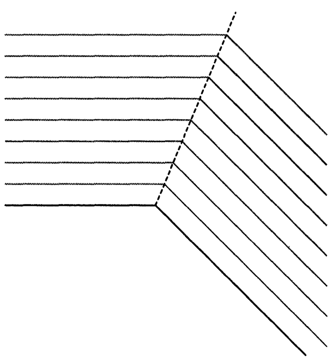

FIG. 3.

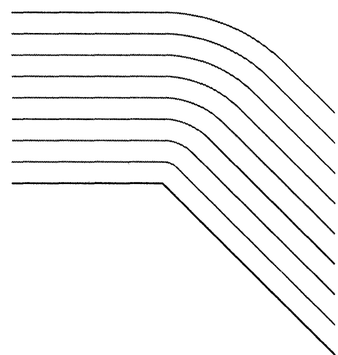

FIG. 4.

displacement of Figure 5. For these simple Θ curves, the theory of Pipkin and Rodgers (based on the fact that the curvature of the normal curves is zero) is sufficient to give a complete description of the kinematically feasible deformations, but for more complicated curves, a more precise method of obtaining a solution is required. It is with our simple conservation law and the theory which is to follow that we can find and describe solutions to any boundary problem (even for nowhere differentiable Θ) and address the issue of nonuniqueness, as exhibited for instance in Figures 3 and 4 (or 5 and 6), from a solely kinematic point of view. If a downward problem is considered, that is, specification of a deformed upper (top) fibre, all the previous statements and diagrams have analogues pertaining to the lower fibre configurations.

In terms of the regularity of solutions under the assumption of smooth boundary displacement curves, we have the following.

PROPOSITION 2.1. *Let Θ be C^2. A solution to* (2.1) *develops a discontinuity d* (*normal distance*) *away from the boundary fibre Θ where d is given by*

$$d = \inf_x \frac{1}{\tilde{\kappa}(x)},$$

and $\tilde{\kappa}(x)$ is the signed curvature of $\Theta(x)$; i.e., $\tilde{\kappa}$ is positive if Θ is concave up and negative if concave down. Hence the fibre which is initially d above the bottom fibre in the undeformed configuration will be either the "first" fibre to have discontinuous slope or the "last" fibre whose slope is continuous. As we shall see, the former corresponds

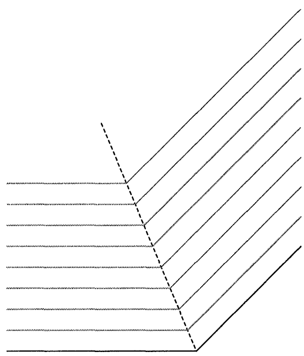

FIG. 5.

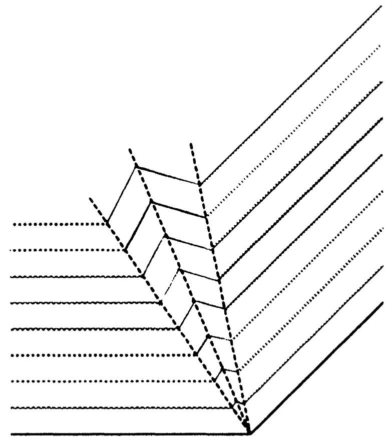

FIG. 6.

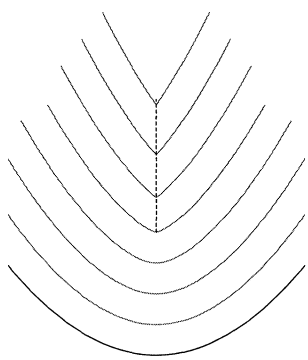

FIG. 7.

to the centre of a compression wave and the latter to a shock generation point (*see* §3 *for definitions*).

Note that if Θ' is always negative then $d < 0$ and a continuous solution u is uniquely determined by Θ. Proposition 2.1 could probably be proved with geometry alone (radius of curvature, etc.). We give an analytical proof similar to the one used to find a precise breakdown time for a solution of the Burgers equation ((1.1) with $f(u) = \frac{1}{2}u^2$). To begin, we need an expression for the curvature of the fibres in terms of our variable u.

LEMMA 2.2. *Let u be a smooth solution to* (2.1). *The curvature of the deformed fibre associated with u through (x, y) is given by*

$$\kappa(x,y) = \frac{1}{1-u^2(x,y)} \left| \frac{\partial u(x,y)}{\partial x} \right|. \tag{2.3}$$

Proof. The fibre through $(x; y)$ has a parametrization $(x(t), y(t))$ where

$$\frac{dx}{dt} = a_1 = \sqrt{1-u^2}, \qquad \frac{dy}{dt} = a_2 = u.$$

The curvature is given by

$$\kappa(x,y) = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{\frac{3}{2}}} = \left| a_1^2 \frac{\partial a_2}{\partial x} + \frac{\partial a_2}{\partial y} a_1 a_2 - \frac{\partial a_1}{\partial x} a_1 a_2 - \frac{\partial a_1}{\partial y} a_2^2 \right|.$$

Using the fact that $\frac{\partial a_1}{\partial x} = \frac{\partial a_2}{\partial x}(\frac{-a_2}{a_1})$ and $\frac{\partial a_1}{\partial y} = \frac{\partial a_2}{\partial y}(\frac{-a_2}{a_1})$ together with (1.2), we obtain

$$\kappa = \left| \frac{\partial a_2}{\partial x} + \frac{\partial a_2}{\partial y} \left(a_1 a_2 + \frac{a_2^3}{a_1} \right) \right| = \left| \frac{\partial a_2}{\partial x} + \frac{\partial a_2}{\partial x} \left(a_2^2 + \frac{a_2^4}{a_1^2} \right) \right|$$
$$= \frac{1}{1-a_2^2} \left| \frac{\partial a_2}{\partial x} \right| = \frac{1}{1-u^2} \left| \frac{\partial u}{\partial x} \right|. \qquad \square$$

Proof of Proposition 2.1. By Lemma 2.2,

$$\kappa(x,y) = \frac{1}{1-u^2} \left| \frac{\partial u}{\partial x} \right| \quad \text{or} \quad \tilde{\kappa} = \frac{1}{1-u^2} \frac{\partial u}{\partial x}.$$

Next we examine the behavior of $\frac{\partial u}{\partial x}$ along characteristic lines. First, note that in general $\frac{\partial u}{\partial x}$ is not defined on the boundary Θ curve. However, we define it as the limit along characteristics. To be precise, for every $(x,y) \in \{ (x,y) : y > \Theta(x) \}$,

$$\frac{\partial u}{\partial x} = (1-u^2)\tilde{\kappa},$$

and hence define

$$\left. \frac{\partial u}{\partial x} \right|_{\Theta(x)} = (1-u^2)\tilde{\kappa} \Big|_{\Theta(x)}.$$

Differentiating (2.1) with respect to x, we obtain

$$\left(\frac{\partial u}{\partial x} \right)_y - \frac{u}{\sqrt{1-u^2}} \left(\frac{\partial u}{\partial x} \right)_x - \frac{1}{(1-u^2)^{\frac{3}{2}}} \left(\frac{\partial u}{\partial x} \right)^2 = 0.$$

Let $w(x(y),y)$ be the value of $\frac{\partial u}{\partial x}$ along a characteristic $x(y)$ emanating from $(\hat{x}, \Theta(\hat{x}))$. The previous equation becomes

$$\frac{dw}{dy} - \frac{1}{(1-u^2)^{\frac{3}{2}}} w^2 = 0. \tag{2.4}$$

The coefficient of w^2 is constant along characteristics and hence (2.4) can be solved explicitly to obtain

$$w(y) = \frac{1}{C - \frac{1}{(1-u^2)^{\frac{3}{2}}} y},$$

where $C := \frac{1}{(1-u^2)\tilde{\kappa}(\hat{x},\Theta(\hat{x}))} + \frac{1}{(1-u^2)^{\frac{3}{2}}} \Theta(\hat{x})$. Thus, w will blow up when

$$y = \Theta(\hat{x}) + (1-u^2)^{\frac{1}{2}} \frac{1}{\tilde{\kappa}(\hat{x}, \Theta(\hat{x}))}.$$

In terms of normal distance, distance along the characteristics, we have blow up at

$$\sqrt{1-u^2}\,\frac{1}{\tilde{\kappa}(\hat{x},\Theta(\hat{x}))}\,\frac{1}{\sqrt{1-u^2}}=\frac{1}{\tilde{\kappa}(\hat{x},\Theta(\hat{x}))}. \qquad \square$$

For certain Θ, we are posed with a uniqueness problem which on the onset seems identical to that for evolution conservation laws, for example the Burgers equation. However, the physical circumstances in which (2.1) arises differ substantially from other known situations in which a conservation law governs behaviour. Taking into account the experience gained within these evolution situations (where entropy is the guiding force), we make the following requirement for a piecewise smooth solution concerning the characteristics and the crimp line: on each side of the crimp line, characteristics emanating from the Θ curve intersect the crimp line. In the context of a time-dependent conservation law, this is the Lax admissibility (or entropy) criterion (cf. [6]). For either the upward or the downward problem stated here, one can easily verify that solutions which do not satisfy this criterion are geometrically unstable: small perturbations of the Θ curves lead to large differences in the deformation, and hence the composite is unstable and likely to snap (see Proposition 4.1). For the upward problem, this condition is equivalent to requiring the solution to satisfy

$$u(x-,y)\le u(x+,y), \tag{2.5}$$

where, for example, $u(x-,y)=\lim_{z\to x^-}u(z,y)$. Note that the opposite inequality should hold for the downward problem. Figures 2, 4, 5, and 7 are examples of this condition when it is satisfied. In the theory of hyperbolic conservation laws, it is well known that even C^∞ Cauchy data may give rise to a solution whose jump discontinuities may accumulate (cf. [1]). Breaking from the class of piecewise smooth solutions, we look for weak solutions with the property that left and right limits exist and satisfy (2.5). Further, let us agree to call discontinuities of such a solution, points where left and right limits differ, shocks.

For a piecewise smooth Θ it would seem reasonable to propose the following formula for an admissible solution, one whose discontinuities are shocks. Let $\Theta(x)$ be continuous and piecewise differentiable. Let $(x,y)\in B^o$. There exists at least one point on the the curve Θ such that the minimum distance from (x,y) to the Θ curve is achieved at this point. If there is more than one such point (minimizers), define u arbitrarily. Otherwise, call the unique minimizing point $(z,\Theta(z))$. Suppose $\Theta'(z)$ exists. Define

$$u(x,y)=\frac{\Theta'(z)}{\sqrt{1+[\Theta'(z)]^2}}.$$

If $\Theta'(z)$ does not exist, let $d(x,y):=-(\frac{x-z}{y-\Theta(z)})$ and define

$$u(x,y)=\frac{d}{\sqrt{1+d^2}}.$$

The last step amounts to constructing a rarefaction wave in the occurrence of a jump (which must be downward) in the slope of the boundary curve. Alternatively, note that the above definition can be replaced with the following. Define

$$u(x,y)=\frac{d}{\sqrt{1+d^2}}\quad\text{with}\quad d(x,y)=-\left(\frac{x-z}{y-\Theta(z)}\right),$$

where $(z, \Theta(z))$ minimizes distance from (x, y) to the Θ curve. This definition is independent of the smoothness of Θ, and $u(z, \Theta(z))$ does not appear explicitly but rather is built implicitly into the formula. The above definition even makes sense for Θ, which is nowhere differentiable: no boundary values here, but as we shall see, the property that classical characteristics are normal to the fibres is preserved. It is not surprising that the proposed solution involves a minimization. Recall the Lax characterization of the entropy solution to (1.1) with data specified at $y = 0$ (cf. [6]). His formula is based on the relationship of (1.1) to the Hamilton–Jacobi equation

$$U_y + f(U_x) = 0, \tag{2.6}$$

whose solutions are rooted in the calculus of variations. One obtains (2.6) from (1.1) by integration with respect to x. We apply Lax's arguments to obtain existence of an admissible solution for any continuous Θ. In our case, the functional which is minimized has a particular geometric significance.

THEOREM 2.3. *Let $\Theta(x)$ be any continuous function and*

$$G(x, y, z) := \frac{-(x-z)}{\sqrt{(x-z)^2 + (y - \theta(z))^2}}.$$

For $(x, y) \in B^o$, define

$$u(x, y) := G(x, y, z), \tag{2.7}$$

where z minimizes $D(x, y, z) := \sqrt{(x-z)^2 + (y - \Theta(z))^2}$. Then on B^o, u satisfies (2.1) in the sense of distributions with all the discontinuities of u being shocks.

Proof. First note that for any (x, y), $D(x, y, z)$ achieves its minimum at some z. Furthermore, for fixed y, let $z(x)$ denote any minimizer of $D(x, y, z)$. Then $z(x)$ is nondecreasing. Thus for all but a countable number of x, $D(x, y, z)$ has a unique minimizer, and hence for each y, $u(x, y)$ is well defined almost everywhere. We define

$$u_n(x, y) = \frac{\int_{-\infty}^{\infty} G(x, y, z)\, e^{-nD(x,y,z)}\, dz}{\int_{-\infty}^{\infty} e^{-nD(x,y,z)}\, dz}$$

and

$$f_n(x, y) = \frac{\int_{-\infty}^{\infty} f(G(x, y, z))\, e^{-nD(x,y,z)}\, dz}{\int_{-\infty}^{\infty} e^{-nD(x,y,z)}\, dz},$$

where $f(u) := \sqrt{1 - u^2}$. At points (x, y), which have unique minimizers, these integrals are well defined, and

$$u(x, y) = \lim_{n \to \infty} u_n(x, y) \qquad \text{and} \qquad f(u(x, y)) = \lim_{n \to \infty} f_n(x, y).$$

In fact, by considering the measurable function $\liminf u_n - \limsup u_n$, we may conclude that the set where u is not well defined (nonunique minimizer) is a Lebesgue measurable set of $\mathbf{R}^2$ and hence, by Fubini's theorem and the fact that for fixed y this set is countable, of measure zero. Thus, the above limits are valid for almost every $(x, y) \in B^o$. We show that for each n, the smooth function u_n actually solves the equation $(u_n)_y + (f_n)_x = 0$ in the usual sense. Define

$$V_n(x, y) := \log \int_{-\infty}^{\infty} e^{-nD(x,y,z)}\, dz.$$

Computing the partial derivatives of V_n, we obtain

$$\frac{\partial V_n}{\partial x} = n\,u_n(x,y) \quad \text{and} \quad \frac{\partial V_n}{\partial y} = -n\,f_n(x,y),$$

and thus $\partial_y u_n(x,y) + \partial_x f_n(x,y) = 0$. The Lebesgue dominated convergence theorem (note that u_n and f_n are bounded) together with the almost everywhere convergence of $u_n(x,y)$ and $f_n(x,y)$ implies that u satisfies (2.1) in the weak, integral sense; i.e.,

$$\int_B u\phi_y + f(u)\phi_x \, dx\, dy = 0 \quad \text{for every } \phi \in C_c^\infty\left(\{(x,y) \,:\, y > \Theta(x)\}\right).$$

Next, we show that at a point of discontinuity (x_1, y_1), the limits $u(x_1+, y_1)$ and $u(x_1-, y_1)$ both exist and $u(x_1+, y_1) \geq u(x_1-, y_1)$. Consider the smallest minimizer z associated with the point (x,y) and let $x_n \to x^-$, with z_n any minimizer associated with (x_n, y). By continuity of $\Theta(x)$, we must have $z_n \to z$, and therefore by (2.7), $u(x-, y)$ exists. An analogous argument implies that $u(x+, y)$ exists, and by definition of u, inequality (2.5) must hold. This completes the proof of Theorem 2.3. However, additional information can be obtained by constructing an inequality analogous to the entropy inequality for evolution equations. To this end, fix y and let

$$C_y := \{x \,:\, (x,y) \in B^o \text{ and there exists no minimizer for } (x,y) \text{ with } y = \Theta(z)\,\}.$$

C_y consists of an at most countable collection of connected components. Choose $a < b$ in one component. There exists $\Delta > 0$ such that for $x_1 \in [a,b]$ we have $(x_1 - \Delta f'(u(x_1,y)), y - \Delta) \in B^o$ where $u(x_1,y)$ is calculated with any minimizer. Now for $x_2 \in [a,b]$ with $x_1 < x_2$ we must have $x_1 - \Delta f'(u(x_1,y)) \leq x_2 - \Delta f'(u(x_2,y))$. Using the mean-value theorem and the fact that $f''(\cdot) \leq -1$, we obtain

$$\frac{u(x_2,y) - u(x_1,y)}{x_2 - x_1} \geq -\frac{1}{\Delta}. \tag{2.8}$$

It is easy to check that on the interior of the complement of C_y, u is either 1 or -1. Thus inequality (2.8) implies that for every y, the decreasing variation of $u(\cdot, y)$ is locally bounded. This implies that the total variation is also locally bounded and hence $u(\cdot, y)$ is locally of bounded variation. In particular, at each x, left and right limits exist and $u(x+, y) \geq u(x-, y)$. □

Theorem 2.3 implies that even if the boundary fibre is deformed so badly that no tangent directions exist, the fibres immediately above have tangent directions almost everywhere. Thus the nonlinearity of the situation may force infinitely smooth boundary displacements to produce discontinuities but also rescues very irregular boundary displacements. Moreover, for smooth boundary displacement perturbations "close" to such an irregular one, the solutions given by Theorem 2.3 are "close" (see Proposition 4.1), justifying on kinematic grounds the significance of such irregular displacements.

In the next section we study the structure of the solution which will aid in demonstrating a certain uniqueness. In order to initiate the study of uniqueness, we must formulate precisely the boundary value problem. Recall that Theorem 2.3 does not explicitly refer to boundary values. The widest class of Θ for which boundary values are meaningful is that of functions locally of bounded variation (thus $\Theta'(x)$ exists a.e.). In addition to u being a weak admissible solution (cf. (2.5)) to (2.1) on B^o, we could require

$$u(x,\Theta(x)) = \frac{\Theta'(x)}{\sqrt{1+[\Theta'(x)]^2}}, \tag{2.9}$$

where the above equality is interpreted in the sense of inner trace (cf. Volpert [11]). This is a sort of measure theoretical limit or equivalent boundary values for a BV (functions of bounded variation) function. A locally integrable function is of class BV (cf. [11]) if its distributional partial derivatives are locally finite Borel measures. This is a multivariable generalization of functions of bounded variation on the line. It turns out that any weak solution whose left and right limits exist and satisfy (2.5) is of class BV.

3. The structure of admissible deformations and generalized characteristics. Having established the existence of an admissible solution (we still need to modify the adjective "admissible"), we now study its structure. The method of generalized characteristics (cf. [1], [2]) is perfect for us, and, in the present setting, (2.1) gives a generic picture of the generalized characteristics and their various properties. As we have seen, the characteristics intersect, forming shocks (crimp lines) in the material. Thus, certainly, at shock points we can only guarantee the existence of left and right fibre directions. This is, of course, the situation in any hyperbolic conservation law and led Dafermos [1], [2] to study the structure of "characteristic" curves of a weak solution with only the property that left and right limits exist and satisfy the appropriate inequality analogous to (2.5). In our context, we know that at least one admissible solution exists and is much more regular than he first assumes. However, this regularity, together with additional structural information, comes from his theory. Dafermos considered characteristics of any admissible solution to an evolution equation defined in the upper half plane. In our case, we have a solution defined above a function curve, and, moreover, if no assumption on the differentiability of Θ is given, characteristic speeds (with respect to y) can be infinite (in the case where $u = \pm 1$). Hence, characteristics can be horizontal, but, as it turns out, shocks are never horizontal. Some results are thus restricted to the solution of Theorem 2.3 for which some a priori information is known. For the study of uniqueness we work within a class of solutions for which fibre directions are never vertical; i.e., $u \neq \pm 1$. Results concerning generalized characteristics which are independent of the above-mentioned differences are stated without proof. These consist of local results on the structure of the genuine characteristics and the shocks. All emitted proofs in this section can be found in either [1] or [2].

To start, let u be a weak solution to (2.1) on B^o such that (s.t.) $u(x\pm, y)$ exists for almost every y and (2.5) holds. Again, to simplify the notation, let $f(u) := \sqrt{1-u^2}$. A Lipschitz continuous curve $\xi(\cdot) : [a,b] \to (-\infty,\infty)$ is said to be a *characteristic* if for almost all $y \in [a,b]$,

$$\dot{\xi}(y) \in [f'(u(\xi(y)+,y)),\, f'(u(\xi(y)-,y))]. \tag{3.1}$$

Corresponding to the case where $u = \pm 1$, we include the possibility that ξ may consist in part of horizontal line segments. Differential equations where the right-hand side is discontinuous have been studied by Filippov (*cf.* [3]/) who, in the above context, has shown that for any $(\bar{x},\bar{y})$ in the interior of B, there exists at least one *forward* (*away from* $y = \Theta(x)$) characteristic defined on some interval and at least one *backward* (*toward* $y = \Theta(x)$) characteristic defined on another interval. The set of forward (or backward) characteristics through $(\bar{x}, \bar{y})$ spans a funnel confined between a minimal and maximal forward (backward) characteristic through $(\bar{x}, \bar{y})$. It may be that the maximal and minimal forward (or backward) characteristics are the same, and in fact this turns out to be the case for the forward characteristics. The maximal and minimal backward characteristics (and part of the forward characteristics) may be horizontal line segments, and it is instructive to think of the adjectives backward and forward as toward and away from the Θ boundary, respectively. For all but uniqueness, we will be concerned with the solution of Theorem 2.3. For uniqueness, where we need to deal with general admissible solutions, we will make the extra assumption that fibre directions are not vertical. Hence, for simplicity of notation, we assume all characteristics can be parametrized by the variable y. By examining directly solutions from Theorem 2.3, we will be able to analyze what we have missed.

Denote the minimal and maximal backward characteristics through $(\bar{x}, \bar{y})$ by

$$\xi_-(y, \bar{x}, \bar{y}) \quad \text{and} \quad \xi_+(y, \bar{x}, \bar{y}),$$

respectively. In view of the substantial freedom associated with (3.1), it would seem that little more could be said about characteristics. However, there is an intimate relationship between u and f; i.e., u is a weak solution of (2.1) and satisfies (2.5). As it turns out, this implies that characteristics must travel at either classical characteristic speed or shock speed (given by the Rankine–Hugoniot jump conditions). Moreover, these results imply that for each y, $u(\cdot, y)$ is Lipschitz continuous as a map from $\mathbf{R}$ to L^1_{loc}, and this, together with an inequality analogous to (2.8), establishes u as a BV function. A characteristic, $\xi(\cdot) : [a, b] \to (-\infty, \infty)$, is called *genuine* if

$$u(\xi(y)+, y) = u(\xi(y)-, y) \quad \text{for almost all} \quad y \in [a, b].$$

We again include as genuine the cases of infinite propagation speed ($u = \pm 1$) where the characteristics are horizontal line segments. Using the assumption (2.5), Dafermos showed that the minimal and maximal backward characteristics are genuine, and, moreover, any genuine characteristic ξ is a classical one in the following sense. There exists a constant $\bar{u}$ such that ξ is a straight line with slope $f'(\bar{u})$ and

$$\bar{u} = u(\xi(y)+, y) = u(\xi(y)-, y) \quad \text{for every} \quad y \in (a, b). \tag{3.2}$$

These results imply that two genuine characteristics may intersect only at their endpoints and that the minimal and maximal backward characteristics $\xi_-(y; \bar{x}, \bar{y})$ and $\xi_+(y; \bar{x}, \bar{y})$ are lines with slope $f'(u(\bar{x}-, \bar{y}))$ and $f'(u(\bar{x}+, \bar{y}))$, respectively. In addition,

$$\begin{aligned} u(\xi_-(y; \bar{x}, \bar{y})\pm, y) &= u(\bar{x}-, \bar{y}), \qquad y \in (\alpha, \bar{y}), \\ u(\xi_+(y; \bar{x}, \bar{y})\pm, y) &= u(\bar{x}+, \bar{y}), \qquad y \in (\sigma, \bar{y}), \end{aligned}$$

where, for example, $\xi_-(y; \bar{x}, \bar{y})$ is defined on $(\alpha, \bar{y})$; i.e., $\Theta\left(\xi_-(\alpha; \bar{x}, \bar{y})\right) = \alpha$. Hence the curves $\xi_-(y; \bar{x}, \bar{y})$ and $\xi_+(y; \bar{x}, \bar{y})$ coincide (are the same) iff $u(\bar{x}-, \bar{y}) = u(\bar{x}+, \bar{y})$.

Next, we direct our attention to the solution of Theorem 2.3.

PROPOSITION 3.1. *Let u be given by (2.7). A backward characteristic emanating from $(\bar{x}, \bar{y})$ is genuine iff it is a line segment from $(\bar{x}, \bar{y})$ to $(z, \Theta(z))$ where $D(\bar{x}, \bar{y}, z)$ is minimum. That is, all genuine backward characteristics are line segments which minimize distance to the Θ boundary curve.*

Proof. The fact that a genuine characteristic ξ must be a line segment to a distance minimizing point point $(z, \Theta(z))$ follows directly from the definition of u and from the fact that ξ is a line with slope $f'(\bar{u})$, $\bar{u}$ given by (3.2). Now suppose χ is a line segment from $(\bar{x}, \bar{y})$ to the $\Theta(x)$ curve which minimizes distance. If χ is horizontal then it is genuine. Otherwise, suppose at some interior point of χ, u had different left and right limits. By the first part of the theorem there would exist two distinct distance-minimizing line segments from the interior point to $\Theta(x)$ (the minimal and maximal backward characteristics). This would contradict the fact that χ was distance minimizing. □

In terms of the forward characteristics, the fact that genuine characteristics can only intersect at their endpoints implies that through each point in B^o there exists a *unique forward characteristic.* In our setting, Proposition 3.1 implies that for any $\Theta(x)$ continuous, the forward characteristic cannot escape (i.e., is defined as $y \to \infty$) nor intersect the Θ boundary. Corresponding to the case of a point where both left and right limits are either 1 or -1, the forward characteristic will start off as a horizontal line segment but eventually will become a curve parametrized by y, which is defined as $y \to \infty$ (i.e., a shock curve; see below).

We now examine points of discontinuity (shocks) of any weak solution satisfying (2.5). For the solution of Theorem 2.3, it should be clear that the set of shocks has a particular geometric significance. Consider a continuous curve given by $y = \Theta(x)$. What points $P \in \{(x, y) : y > \Theta(x)\}$ have the following property: there exists more than one point on the Θ curve which minimizes distance from P to the curve. Returning to the general situation pertaining to any admissible solution, it turns out that the points of discontinuity lie on curves which propagate without bound through the material and can always be parametrized by y. A characteristic $\eta(\cdot)$ defined on $[\bar{y}, \infty)$ is called a *shock curve* if

$$u(\eta(y)-, y) < u(\eta(y)+, y) \qquad \text{for all} \quad y \in (\bar{y}, \infty).$$

Two particular types of points are relevant to the shock curves. A point $(\bar{x}, \bar{y}) \in B^o$ is called a *shock generation point* if the (unique) forward characteristic through it is a shock curve and there is only one (genuine) backward characteristic. A point $(\bar{x}, \bar{y}) \in B^o$ is called a *centre of a centred compression wave* if there are two distinct minimal and maximal backward characteristics and every other backward characteristic is also genuine. The funnel confined between the minimal and maximal backward characteristics stemming from a centre of a centred compression wave is filled by distance-minimizing lines, and hence the bottom boundary of this funnel (the Θ part) consists of a circular arc. Note that the shock generation points (not shocks by our definition) are included in the shock curves.

We state some results of Dafermos pertaining to the shocks. If at some point $(\bar{x}, \bar{y})$, $u(\bar{x}-, \bar{y}) < u(\bar{x}+, \bar{y})$, then the unique forward characteristic $\eta(\cdot)$ has the property that for all $y \in [\bar{y}, \infty)$, $u(\eta(y)-, y) < u(\eta(y)+, y)$ (i.e., is a shock curve). The following regularity properties of the solution and shock curve hold. Let $\eta(\cdot)$ be a shock curve defined on $[\bar{y}, \infty)$ and let $w_-(y) = u(\eta(y)-, y)$, $w_+(y) = u(\eta(y)+, y)$. Then $w_\pm(y)$ is continuous from the right on $[\bar{y}, \infty)$, and, further, limits $w_\pm(y-)$ from the left exist for all $y \in [\bar{y}, \infty)$ and $w_-(y-) \le w_-(y)$, $w_+(y-) \ge w_+(y)$. Additionally, $\dot{\eta}(y)$ exists and is continuous except on the at most countable set of interaction points of η with other shock curves or centres of centred compression waves. Moreover, for a.e. $y \in [\bar{y}, \infty)$,

$$\dot{\eta}(y) = \frac{f(u(\eta(y)+, y)) - f(u(\eta(y)-, y))}{u(\eta(y)+, y) - u(\eta(y)-, y)}. \tag{3.3}$$

Thus all the discontinuities in the fibre direction lie on Lipschitz curves which are smooth almost everywhere: the same situation as with the piecewise smooth solutions given in Figures 2 through 7. Moreover, (3.3) combined with the fact that for any y the set of discontinuities of $u(\cdot, y)$ is at most countable implies that there are at most countably many shock curves. These shock curves (crimp lines) bisect the fibres at their points of discontinuous slope. Alternatively, the shock curve bisects the angle between the minimal and the maximal backward characteristics.

The following picture of backward characteristics emerges for the solution of Theorem 2.3 which as we shall see (Propositions 3.3 and 3.4), at least for certain Θ, is unique. For any $P = (\bar{x}, \bar{y}) \in B^o$ consider $M = \{z : D(\bar{x}, \bar{y}, z) \text{ is minimum}\}$. The backward characteristics lie between lines joining $(\bar{x}, \bar{y})$ to $\Theta(x)$ at $x = \min M$ and $x = \max M$. In-between characteristics, which fill up the region, are either distance-minimizing line segments, shock curves, or combinations of the two. If $\min M \neq \max M$, the unique forward characteristic through P is a shock curve. This is illustrated in Figure 8, where the dotted line indicates the shock curve beginning at a shock generation point and ending at P.

If $\min M = \max M$ and the unique distance-minimizing line segment was horizontal and to the left of P (i.e., $u(P) = -1$), then the unique forward characteristic would start off as a horizontal line ($y = \bar{y}$) to the right of P and continue until it reaches a point where another minimizer exists (this must happen), after which it becomes a shock and propagates upward. If M contains more than two points, then P is either a point of shock curve interaction or a centre of a compression wave. Thus the set of points in B which have more than two distance minimizers on $y = \Theta(x)$ is at most countable. We have proved Theorem 3.2.

THEOREM 3.2. *Let $\Theta(x)$ be any continuous function and consider the planar curve given by $y = \Theta(x)$. The set of points above (below) the curve which have the property that their minimum distance to the curve is attained at more than one point consists of a countable number of Lipschitz curves which are smooth almost everywhere. As y increases (decreases) they either merge with another such curve or are unbounded above (below). Further, the set of points which have more than two minimizers is at most countable and is contained in the union of the interaction points and the finite endpoints of these Lipschitz curves.*

As for continuity of the fibre direction, we have the following. Let $(\bar{x}, \bar{y}) \in B^o$, $u(\bar{x}\pm, \bar{y}) \neq \pm 1$, and η be the unique forward characteristic emanating from $(\bar{x}, \bar{y})$.

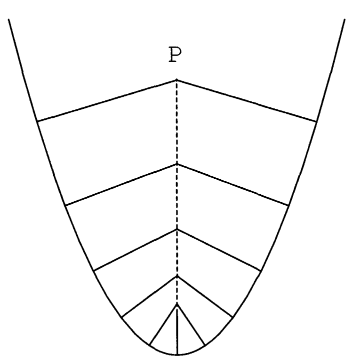

FIG. 8.

Then $(\bar{x}, \bar{y})$ is a point of continuity of u relative to the sets

$$\{(x, y) : y \geq \bar{y}\,,\ x \leq \eta(y) \text{ or } y < \bar{y}\,,\ x \leq \xi_-(y; \bar{x}, \bar{y})\},$$
$$\{(x, y) : y \geq \bar{y}\,,\ x > \eta(y) \text{ or } y < \bar{y}\,,\ x \geq \xi_+(y; \bar{x}, \bar{y})\},$$

with the respective limits being $u(\bar{x}-, \bar{y})$ and $u(\bar{x}+, \bar{y})$. If $u(\bar{x}-, \bar{y}) = -1$ and $u(\bar{x}+, \bar{y}) > -1$ or the analogous statement for $+1$, then the above have appropriate adjustments. If we have $u(\bar{x}\pm, \bar{y}) = -1$ (or 1), then we can directly check that u is continuous in a neighbourhood of $(\bar{x}, \bar{y})$. Thus the solution $u(x, y)$ is continuous at $(\bar{x}, \bar{y})$ iff $u(\bar{x}-, \bar{y}) = u(\bar{x}+, \bar{y})$. We also have continuity on either side of the shock curve. Precisely, let η be a shock curve and let $\bar{y}$ be a point of continuity of the functions $w_\pm(y) = u(\eta(y)\pm, y)$. Then $(\eta(\bar{y}), \bar{y})$ is a point of continuity of $u(x, y)$ relative to the sets $\{(x, y) : x < \eta(y)\}$ and $\{(x, y) : x > \eta(y)\}$ with the respective limits being $u(\eta(\bar{y})-, \bar{y})$ and $u(\eta(\bar{y})+, \bar{y})$. So for any continuous lower fibre displacement Θ, the resulting fibres are smooth on the complement of the shock set with the fibre directions continuous in x and y.

We now direct our attention toward uniqueness. As mentioned at the end of §2, we work with the assumption that Θ is locally of bounded variation. In this case we have sufficient regularity in terms of both the solution and the boundary of the domain B to invoke the Gauss–Green theorem (see [11]). Moreover, assume further that $\Theta'(x)$ is bounded. This precludes the possibility that $u = \pm 1$ for u given by (2.7) and also insures the integrability of the trace on $y = \Theta(x)$. We show that the fibre configuration associated with this solution u is unique amongst all configurations where fibre directions are never vertical and which satisfy the compatibility relation with the boundary Θ, i.e., (2.9).

PROPOSITION 3.3. *Let Θ be locally of bounded variation and $\Theta'(x)$ be bounded. Let v be any weak solution to* (2.1) *such that $v \neq \pm 1$ and v satisfies* (2.5) *and* (2.9) *in the sense of trace. Then genuine characteristics are distance-minimizing line segments.*

Proof. The fact that genuine backward characteristics cannot be horizontal implies that they are defined down to the $y = \Theta(x)$ curve. Let $\xi(\cdot)$ be a genuine backward characteristic through $(\bar{x}, \bar{y})$ and let $\chi(\cdot)$ be any line from $(\bar{x}, \bar{y})$ to a point on $y = \Theta(x)$. We show that the length of ξ (stopping of course when ξ hits Θ) is no greater than the length of χ. Let R be the "triangular" region which ξ and χ make with Θ. We use the Gauss–Green theorem which is valid for our BV solution on a set whose characteristic function is in turn of bounded variation; this is where we need the hypothesis on $\Theta(x)$. Integrating (2.1),

$$\begin{aligned} 0 &= \int\!\!\int_R \operatorname{div}_{(x,y)} (f(v), v)\, dx\, dy \\ &= \int_\xi (f(v), v) \cdot \mathbf{n}\, dH^1 + \int_\chi (f(v), v) \cdot \mathbf{n}\, dH^1 + \int_{\Theta\text{part}} (f(v), v) \cdot \mathbf{n}\, dH^1, \end{aligned}$$

where H^1 denotes one-dimensional Hausdorff measure. Since (2.9) holds in the sense of trace and $\mathbf{n}$ is perpendicular to Θ a.e., the last integral is zero. The theory of generalized characteristics implies that ξ is a line segment with slope $f'(\bar{u})$, $\bar{u}$ given by (3.2). This combined with the Schwarz inequality implies

$$|\xi| = \left| \int_\xi (f(v), v) \cdot \mathbf{n}\, dH^1 \right| = \left| \int_\chi (f(v), v) \cdot \mathbf{n}\, dH^1 \right| \leq |\chi|. \qquad \square$$

Proposition 3.3 implies there is at most one admissible solution which satisfies (2.9) in the sense of trace. In order to establish that the solution constructed in Theorem 2.3 is the unique solution to the boundary value problem we need the following.

PROPOSITION 3.4. *Let Θ be as in Proposition* 3.3. *Then the solution constructed in Theorem* 2.3 *satisfies* (2.9) *in the sense of trace.*

Proof. Let P_1 be any point in B and let ν be a genuine backward characteristic from P_1. Consider the fibre F through P_1 (say in the direction of increasing x), that is, the continuous solution to the ordinary differential equations (2.2) . Let P_2 be any point on F and δ a genuine backward characteristic emanating from P_2. The fact that fibres (trajectories of (2.2)) are normal characteristics implies that the length of ν equals the length of δ. Integrating (2.1) in the region bounded by $\Theta(x), F, \nu$, and δ, using the flexibility in choosing P_1 and P_2, and the Gauss–Green theorem, we obtain

$$\int_\gamma (f(u), u) \cdot \mathbf{n}\, dH^1 = 0,$$

where the integrand involves the trace of u on the boundary $\Theta(x)$, $\mathbf{n}$ is the upper normal to $y = \Theta(x)$, and γ is any bounded and connected subset of $\{(x, y) : y = \Theta(x)\}$. Hence, the same equality holds true for any measurable subset of $y = \Theta(x)$ and the result follows. $\square$

Next we address the case of smooth Θ curves and the resulting regularity of the fibres. Let $\Theta(x) \in C^{k+1}$ with $k \geq 1$. We note that $f(u) = \sqrt{1-u^2}$ is C^∞ except when $u = \pm 1$. However, since $\Theta'(x)$ exists for all x and is continuous, the solution u given by Theorem 2.3 takes values strictly between -1 and 1. In fact, the solution must satisfy $u(x, \Theta(x)) = \frac{\Theta'(x)}{\sqrt{1+(\Theta'(x))^2}} =: w(x)$. Now let $z^\pm$ be the interceptors of $\xi_+(\bar{x}, \bar{y})$ and $\xi_-(\bar{x}, \bar{y})$ with the $y = \Theta(x)$ curve. The properties of the minimal and maximal backward characteristics imply that

$$\frac{\bar{x} - z^\pm}{\bar{y} - \Theta(z^\pm)} = f'\left(w(z^\pm)\right) = -\Theta'(z^\pm)$$

or

$$\bar{x} = z^\pm - \left(\bar{y} - \Theta(z^\pm)\right)\Theta'(z^\pm) \tag{3.4}$$

and

$$u(\bar{x}\pm, \bar{y}) = w(z^\pm). \tag{3.5}$$

Recall from the proof of Theorem 2.3 that, for fixed $\bar{y}$, $z^\pm$ are increasing functions of $\bar{x}$ and are continuous from the left and right. For our smooth Θ, they are in fact strictly increasing. Hence for every $(\bar{x}, \bar{y}) \in B^o$, differentiating (3.4) with respect to z^+ (z^-) gives

$$1 - (\bar{y} - \Theta(z))\Theta''(z) + (\Theta'(z))^2 \geq 0, \qquad z = z^+ \text{ or } z^-. \tag{3.6}$$

At a point of continuity $(\bar{x}, \bar{y})$, z is a continuous function of x and hence by the implicit-function theorem we have Proposition 3.5.

PROPOSITION 3.5. *u is C^k on a neighbourhood of a point of continuity $(\bar{x}, \bar{y})$ as long as*

$$1 - (\bar{y} - \Theta(z))\Theta''(z) + (\Theta'(z))^2 > 0. \tag{3.7}$$

If the above is zero then $(\bar{x}, \bar{y})$ *is the centre of the osculating circle of* $y = \Theta(x)$ *at* $x = z$.

If equality holds in (3.6), $(\bar{x}, \bar{y})$ is either a shock generation point or centre of a compression wave. If $(\bar{x}, \bar{y})$ is a point of continuity of u, then $(\bar{x}, \bar{y})$ is a shock generation point. In either case one can directly check that $(\bar{x}, \bar{y})$ is the centre of the osculating circle (also known as the circle of curvature) for $y = \Theta(x)$ at $x = z$. Moreover the forward characteristic through $(\bar{x}, \bar{y})$ is a shock and hence $(\bar{x}, \bar{y})$ is part of a shock curve. On the complement of the union of shock curves (the set of shocks together with the shock generation points), (3.6) must hold as a strict inequality. Thus by Proposition 3.5, the complement of the union of shock curves is open and u is C^k on this set. In terms of smoothness of the shock curves, the previous results concerning continuity on either side of a shock curve, (3.3), (3.4), (3.5), and the implicit-function theorem imply the following. If $(\bar{x}, \bar{y})$ is a point on a shock curve $\eta(\cdot)$ at which (3.7) holds and $\bar{y}$ is a point of continuity of $u(\eta(y)\pm, y)$, then $\eta(\cdot)$ is C^{k+1} smooth on a neighbourhood of $\bar{y}$. Further, on a neighbourhood of $(\bar{x}, \bar{y})$, $u(x, y)$ is C^k smooth on either side of $\eta(\cdot)$.

We close this section with a simple example of a smooth Θ, which may give rise to a solution that is not piecewise smooth. A solution is said to be piecewise smooth if any bounded subset of B intersects with only a finite number of shock curves. Failure to satisfy this condition is a consequence of an accumulation of shock generation points which, in view of the geometry of the problem, happens only if there exists an accumulation of inflection points in $\Theta(x)$. Consider

$$\Theta(x) = x^k \sin\left(\frac{1}{x}\right) \text{ for } x \geq 0 \text{ and } \Theta(x) = -|x|^k \sin\left(\frac{1}{x}\right) \text{ for } x < 0.$$

For $x > 0$, the signed curvature is given by $\tilde{\kappa}(x) = \frac{\Theta''(x)}{1+(\Theta'(x))^2}$. The numerator is

$$\left(k(k-1)x^{k-2} - x^{k-4}\right)\sin\frac{1}{x} - \left((2k-2)x^{k-3}\right)\cos\frac{1}{x}.$$

Suppose $k > 4$. In this case $\tilde{\kappa} \to 0$ as $x \to 0$ and hence, by Proposition 2.1, the shock generation points do not accumulate or, rather, accumulate at infinity. Suppose $k < 4$ (and $k \geq 2$). Then $\sup \tilde{\kappa}$ over a quasi period of Θ with centre x approaches ∞ as $x \to 0$, and thus, by Proposition 2.1, the shock generation points accumulate at the origin. Finally, let us consider the critical case of $k = 4$. In this case, $\sup \tilde{\kappa}$ over a quasi period of Θ with centre x approaches 1 as $x \to 0$, and thus shock generation points will accumulate in the interior of the material, in fact at the point $(0, 1)$, and the solution is not piecewise smooth. In all cases, the shock curves eventually become the y-axis, which remains a shock as $y \to \infty$, albeit a weaker and weaker one.

4. Admissibility. Via the admissibility condition (2.5), we have shown existence for general boundary displacements and uniqueness for certain reasonable boundary displacements. As previously noted, ignoring condition (2.5) implies that an infinite number of deformations exist (see Figure 6). How can one justify that the deformation obtained in Theorem 2.3 is the one which is likely to occur without going to the level of the constitutive behavior of the materials in question and the forces which accompany the deformation? A mechanist may debate whether this is possible at all; in fact Pipkin and Rodgers concluded that it is not (cf. [9]). We show with kinematics alone that this class of solutions exhibits certain desirable properties which other solutions fail to exhibit.

4.1. Stability. The first such property is stability with respect to perturbation of the boundary fibre displacement. Consider Figures 3 and 4. Both are kinematically feasible deformations for the particular boundary displacement problem. It is not hard to see that the deformation of Figure 3 fails to satisfy the following: under smooth perturbations of $\Theta(x)$ there are associated, kinematically feasible deformations which converge to the deformation associated with $\Theta(x)$. This is satisfied by the solution in Figure 4. This type of stability fails when characteristics on either side of a crimp line do not, as they propagate away from the specified boundary fibre, intersect the crimp line. With the aid of the previous minimization formula (2.7) and Proposition 3.1, we make these comments precise.

PROPOSITION 4.1. *Let $\Theta_n(x)$ be continuous functions which converge uniformly to $\Theta(x)$, and let u_n, u be the solutions, via* (2.7), *for Θ_n and Θ, respectively. If $(\bar{x},\bar{y})$ is a point of continuity of u, then*

$$u_n(\bar{x}\pm,\bar{y}) \to u(\bar{x},\bar{y}).$$

Proof. Let $z_n^{\pm}$ be the interceptors with $y=\Theta_n(x)$ of the minimal and maximal backward characteristics through $(\bar{x},\bar{y})$ associated with u_n and z the interceptor with $y=\Theta(x)$ of the unique backward characteristic associated with u. By Proposition 3.1, $D_n(\bar{x},\bar{y},z_n^{\pm})$ is the minimum distance from $(\bar{x},\bar{y})$ to $y=\Theta_n(x)$ and $D(\bar{x},\bar{y},z)$ is the minimum distance from $(\bar{x},\bar{y})$ to $y=\Theta(x)$. The fact that z is unique and $\Theta_n(x)$ converges uniformly to $\Theta(x)$ implies $z_n^{\pm}\to z$. The result now follows from (2.7). $\square$

4.2. "Viscosity". It is well known that one way to construct solutions to (1.1) with $f''>0$ (or <0) is via vanishing viscosity, limits of smooth solutions to the parabolic equation

$$\partial_{x_1}u+\partial_{x_2}[f(u)]=\epsilon\frac{\partial^2 u}{\partial x_2^2} \tag{4.1}$$

(cf. Oleinik [7]). Moreover, the limits are known to correspond to physically admissible solutions which, in the context of an evolution equation (1.1), entail satisfying the entropy inequality. Physically, (4.1) corresponds to the introduction of small viscosity, thermal conductivity, or some other frictional effect which brings in second derivatives. Analytically, we cannot blindly replace (2.1) with a second-order equation of the form (4.1) if we assume the "viscous" equation will hold for both the upward and the downward problems. Such equations have solutions in one direction alone: the relationship between the sign of the coefficient of the first term on the left and the term on the right is critical. Without further speculation, we must consider the physical situations surrounding the introduction of the second-order term. As a first attempt we provide a simple model for constructing a viscous equation by the relaxation of the inextensibility constraint. For the rest of §4.2 let Θ be piecewise C^2. Assuming the composite is incompressible (itself an idealization), one would expect that the fibres, though creating anisotropic behavior in the composite, permit small changes in length and that this ability should enable them to deform within the composite without the formation of discontinuities: crimp lines. The idea is that in implementing a given boundary displacement we have control only of the length of the boundary fibre and, hence, no contraction nor extension occurs on it. However, the other fibres will be free to change their length slightly and with the amount of change proportional to the curvature of the fibres. That is, we assume the following relationship between tangent vector components:

$$a_1 = \sqrt{1 - a_2^2} + \alpha\epsilon\kappa, \tag{4.2}$$

where $\alpha = \pm 1$ depending on whether the deformation causes fibre compression or extension and κ is the curvature. We proceed formally to derive the "viscous" equations, localize, and look for viscous profiles about jump discontinuities in u.

Eventually (cf. (4.4)) terms of order ϵ^2 will be ignored, and hence we use Proposition 2.1, which assumes inextensibility, to compute the curvature. The curvature of the fibre through (x, y) is thus $\frac{1}{1-u^2}\left|\frac{\partial u}{\partial x}\right|$. Assuming u on the boundary Θ curve is bounded away from 1, the curvature equals $\left|\frac{\partial u}{\partial x}\right|$ multiplied by a bounded term. For simplicity we replace κ in (4.2) with $\left|\frac{\partial u}{\partial x}\right|$. In order to determine α, one must decide whether the distance between a given pair of fibres increases or decreases as a result of the deformation, that is, the resulting reaction of the matrix to the fibres. An increase in distance will lead to contraction of the fibres, whereas a decrease leads to extension. Consider the "Riemann problem" of boundary displacements which consist of two joined lines, one horizontal (cf. Figures 3, 4, and 5). The distance between fibres will increase if the boundary curve is deformed into the material (cf. Figure 5) and decrease if the boundary curve is deformed away from the material (cf. Figure 3 or 4). The amount of extension/compression will depend on the magnitude of the normal force components which we assume is proportional to the curvature of the fibres. Suppose our specified displacement fibre is the bottom fibre (the upward problem). In this case if u $(= a_2)$ increases with x ($\frac{\partial u}{\partial x}$ is positive), the boundary displacement is "into" the material, whereas if u decreases ($\frac{\partial u}{\partial x}$ is negative), the displacement is "away" from the material. Thus for the upward problem we propose that

$$\alpha\left|\frac{\partial u}{\partial x}\right| = -\frac{\partial u}{\partial x}. \tag{4.3}$$

For the downward problem, the right-hand side of (4.3) should be $+\frac{\partial u}{\partial x}$. Combining these results with (4.2) and (1.2), which holds for any incompressible composite, we obtain

$$\partial_y u + \partial_x[\sqrt{1-u^2}] = \epsilon\frac{\partial^2 u}{\partial x^2} \tag{4.4}$$

for the upward problem and

$$\partial_y u + \partial_x[\sqrt{1-u^2}] = -\epsilon\frac{\partial^2 u}{\partial x^2} \tag{4.5}$$

for the downward problem.

PROPOSITION 4.2. *For* (4.4), *we obtain viscous profiles for the constant states* u_+ *and* u_- *iff* $u_+ > u_-$. *For* (4.5), *we obtain viscous profiles iff* $u_+ < u_-$.

Thus the deformations in Figures 4 and 5 are chosen over the deformations in Figures 3 and 6, respectively.

Proof. The proof follows a standard argument in dynamical systems. In the language of conservation laws, we demonstrate the viscosity shock admissibility criterion. Consider a solution u which is constant on either side of a shock curve $x = s\,y$ where s is the speed of the shock curve. Suppose $u = u_-$ if $x < sy$ and $u = u_+$ if $x > sy$.

We look for conditions on u_- and u_+ such that there exist traveling wave solutions of (4.4) which converge to u. Letting $\zeta = \frac{x-sy}{\epsilon}$, we look for solutions of the form $u(x,y) = \bar{u}(\zeta)$ such that

$$\bar{u}(\zeta) \to \begin{cases} u_- & \text{if } \zeta \to -\infty, \\ u_+ & \text{if } \zeta \to +\infty. \end{cases}$$

Analysis of the phase portrait for the ordinary differential equation that $\bar{u}$ must satisfy yields the appropriate inequality. □

One might justify analytically that the inclusion of the second-order term does indeed produce smooth solutions, assuming no extension/contraction along the boundary fibre, but interior fibres do change their length. According to (4.2), a smooth field of fibre directions exists compatible with the incompressibility constraint. Precisely, let Θ be a piecewise smooth curve and $\epsilon > 0$. There exists a smooth solution to (4.4) on $\{ (x,y) \,:\, y \geq \Theta(x) \}$ and a smooth solution to (4.5) on $\{ (x,y) \,:\, y \leq \Theta(x) \}$, both satisfying

$$u(x, \Theta(x)) = \frac{\Theta'(x)}{\sqrt{1 + [\Theta'(x)]^2}}.$$

If $\Theta(x)$ is constant, this follows from the classical theory (cf. Oleinik [7]). We do not include a proof but refer the reader to §5 for a few comments on a possible proof.

The previous arguments are only a first attempt at bringing in so-called viscous terms and give way to several criticisms. Indeed, the sign of the viscous term in either (4.4) or (4.5) should be related to the local configuration of the fibres (curvature, stresses, etc.) and not to the boundary conditions. We have given a naive argument which, without looking at constitutive properties of the material, attempts to relate these local configurations to the boundary displacement, and, in particular, our arguments apply only to a local neighbourhood of the boundary fibre. A firmer approach to the problem of selecting those deformations which are limits of deformations associated with relaxed problems could be via the study of homogenization of the linearized equations of equilibrium, the Navier equations (see Gurtin [4]) corresponding to a fixed volume fraction of very thin hard fibres in an incompressible matrix. The parameters in these equations (Young and bulk modulus, etc.) should be related to the relaxation of the inextensibility and incompressibility constraints and would tend to infinity (thus producing small terms in the equations) in the idealized, limiting case considered in this paper. Such analysis certainly warrants future attention but is not pursued here.

4.3. Decay. It is straightforward to check that admissible deformations have the following decay properties which other solutions fail to exhibit.

PROPOSITION 4.3. *Let $\Theta(x)$ be continuous with* $\sup \Theta(x) - \inf \Theta(x) < C$ *for some* $C > 0$. *Then*

$$|u(x,y)| = \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{y}}}\right) \quad \textit{as } \mathrm{y} \to \infty.$$

Proof. From (2.7), $|u(x,y)| \leq \frac{|x-z|}{|y-\Theta(z)|}$. Moreover,

$$(D(x,y,z))^2 = (x-z)^2 + (y - \Theta(z))^2 = \left((y-\Theta(z)) + D(x,y,z) - (y-\Theta(z))\right)^2$$
$$= (y-\Theta(z))^2 + \left(D(x,y,z) - (y-\Theta(z))\right)^2 + 2(y-\Theta(z))\left(D(x,y,z) - (y-\Theta(z))\right).$$

By assumption, $D(x, y, z) - (y - \Theta(z)) \leq C$, and hence

$$|x - z| \leq C\sqrt{2(y - \Theta(z)) + C} \quad \text{which implies} \quad |u(x, y)| \leq \frac{C\sqrt{2(y - \Theta(z)) + C}}{y - \Theta(z)}.$$

The result now follows from the fact that $\Theta(z)$ is bounded. □

We gain more rapid decay for data which in addition to being bounded are periodic.

PROPOSITION 4.4. *Let $\Theta(x)$ be continuous and periodic. Then*

$$|u(x, y)| = \mathrm{O}\left(\frac{1}{\mathrm{y}}\right) \quad as\ \mathrm{y} \to \infty.$$

Proof. Let P be the period of Θ. We must have $|x - z| \leq P$. Hence

$$|u(x, y)| \leq \frac{P}{y - \Theta(z)},$$

and the result follows. □

In either of the previous cases, with respect to y, the slopes of the shock curves tend to zero as they propagate away from the specified boundary. Kinematically feasible deformations not in this class do not decay for appropriate Θ. See Figures 9 through 12. Figures 10 and 12 are the unique admissible solutions given by (2.7). It

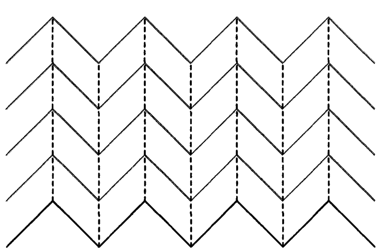

FIG. 9.

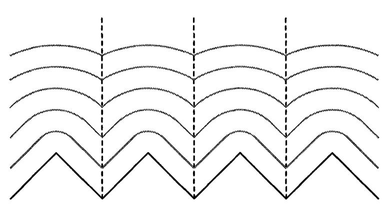

FIG. 10.

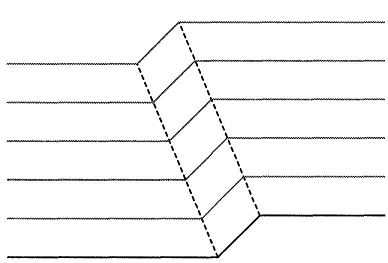

FIG. 11.

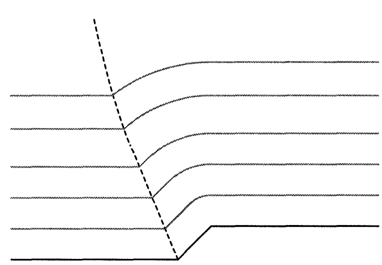

FIG. 12.

is the failure of the characteristics to collide into the crimp lines which precipitates nonzero information to propagate indefinitely.

5. Remarks. To establish the existence of the "viscous" equation (4.4), we could begin by obtaining a local solution. Direct calculation shows that the data curve $y = \Theta(x)$ is characteristic for (4.4) iff $\Theta'(x) = 0$, and hence if $\Theta'(x) \neq 0$ we need to specify an extra boundary condition. The natural condition is

$$\text{the normal derivative of } u \Big|_{y=\Theta(x)} = \epsilon \frac{d\tilde{\kappa}(x)}{dx},$$

where $\tilde{\kappa}(x)$ is the signed curvature of $\Theta(x)$. Once a local solution is found, we can use the parabolicity of the equation to show that the solution can be extended as $y \to \infty$ without blowup (cf. Oleinik [7]). Differentiating (4.4), we obtain an equation for $\partial_x u$ to which we apply the maximum principle for parabolic operators. The relative sign in (4.4) of the viscosity coefficient and the coefficient of $\partial_x u$ is now critical.

Rather different work of Pipkin [8] attempts to alleviate the nonuniqueness for a particular problem via energy minimization with particular constitutive relations. A block (cf. Figure 1) is deformed via loading at some point on the lower boundary, to arrive at two kinematically admissible deformations (similar to Figures 3 and 4 reflected in the x-axis) which roughly share the same upper boundary fibre configuration. It is shown that the deformation with a rarefaction wave (cf. Figure 4) minimizes energy, but the one with the crimp line (cf. Figure 3) fails to minimize energy and moreover does not even correspond to stationary energy.

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