The Relative Pure-Entire Factorization for Geometric Morphisms

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Abstract

A locale $A$ in a topos $\mathcal{E}$ is said to be a Stone locale if it is a compact and zero-dimensional locale. An equivalent description says that $A$ is the locale of ideals of a Boolean algebra in $\mathcal{E}$. A geometric morphism $\varphi : \mathcal{F} \to \mathcal{E}$ is called entire (respect. pure) if $\varphi$ is localic and defined by a Stone locale (respect. if $\varphi_* (2_{\mathcal{F}}) \cong (2_{\mathcal{E}})$, where $2 = 1 + 1$). In [P.T. Johnstone, Factorization Theorems for Geometric Morphisms II, Categorical aspects of Topology and Analysis, Springer, LNM 015 (1982) 216-233] it is shown that every geometric morphism $\varphi$ admits a unique factorization $\varphi \cong \psi \cdot \pi$ where $\psi$ is entire and $\pi$ is pure.

Suppose now that there is a base topos $\mathcal{S}$ over which the toposes are defined and that we only consider geometric morphisms “over $\mathcal{S}$”. Also suppose that instead of $2 = 1 + 1$ in the above, we take the object $\Omega_{\mathcal{S}}$ of truth-values in the topos $\mathcal{S}$. The question that we answer here is the following: under what conditions on $\varphi : \mathcal{F} \to \mathcal{E}$ (over $\mathcal{S}$) does one obtain a relativized version of the pure-entire factorization mentioned above. In the process of answering it in [M. Bunge, J. Funk, M. Jibladze, T. Streicher, Relative Stone Locales, in preparation], we encounter several interesting versions of well-known notions, constructive versions of classically known results, and new problems.