

Applications of anafunctors II

M. Makkai / Nov 12, 2014

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Applications of anafunctors II.

M. Makkai

①

§1

With Ross Street (Cahiers TDC, 1980), we denote by $\underline{\text{Hom}}$ the category (!) of small bicategories and homomorphisms (= strong morphisms) ("small" is relative to an arbitrary choice of a Grothendieck universe)

Theorem $\underline{\text{Hom}}$ is a dual-regular category.

(NB. For 'dual-regular', see "An application of anafunctors (July 7, 2014)", p. ③ (ref. [J])

I repeat the definition:

Definition Category A is dual-regular if there exists a small regular category R such that

$$A \simeq \text{Reg}(R, \text{Set})$$

↑
category equivalence

here, $\text{Reg}(R, \text{Set})$ is the category of all regular functors $R \rightarrow \text{Set}$; $\text{Reg}(R, \text{Set})$ is a full subcategory of $[R, \text{Set}]$.

(VBNB) In the definition, the word 'small' is coordinated with the definition of Set: the latter is the category of small sets. Of the category $\mathcal{A}$, the initial assumption is merely that it be locally small: all hom-sets $\mathcal{A}(A, B)$ are to be small.

The proof of the theorem will be given by constructing two further categories and (forgetful) functors shown next:

$$\begin{array}{ccc}
 & \text{split Sana Bicat} = \text{SSB} & \\
 S \swarrow & & \searrow R \\
 \text{SB} = \text{Sana Bicat} & & \text{Hom} ;
 \end{array}$$

and proving that:

(*) R and S are both full and faithful and surjective on objects;

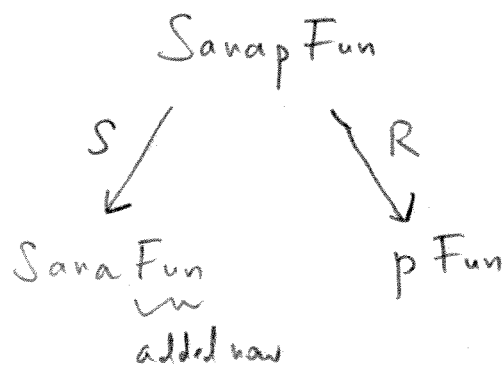
and pointing out the essentially obvious fact that

(**) Sana Bicat is dual-regular.

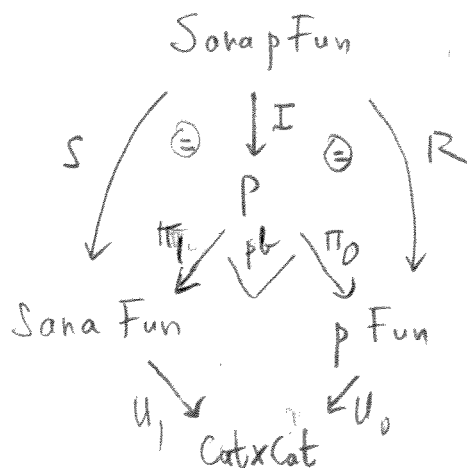
Further preliminary remarks:

The present write-up continues and applies
 "An application of anafunctors" (July 7, 2014)
 by the same author. However, time is a

In that paper underlying the work there,
 we have the span:



see p. ⑥, loc. cit. Let me clarify the construction
 there, and also here, by expanding the spans
 into bigger diagrams: we have, in the previous paper:



P here is the pullback of the two forgetful functors U_0, U_1 :

$$(\Phi, X, A) \xrightarrow{U_0} (X, A)$$

$$(|F|, X, A, \varphi, \psi) \xrightarrow{U_1} (X, A)$$

(with references to the notation in "An application...")

and I the forgetful functor

$$(|F|, X, A, \Phi, \varphi, \psi, \mu) \xrightarrow{I} (|F|, X, A, \Phi, \varphi, \psi)$$

↑
"forget μ "

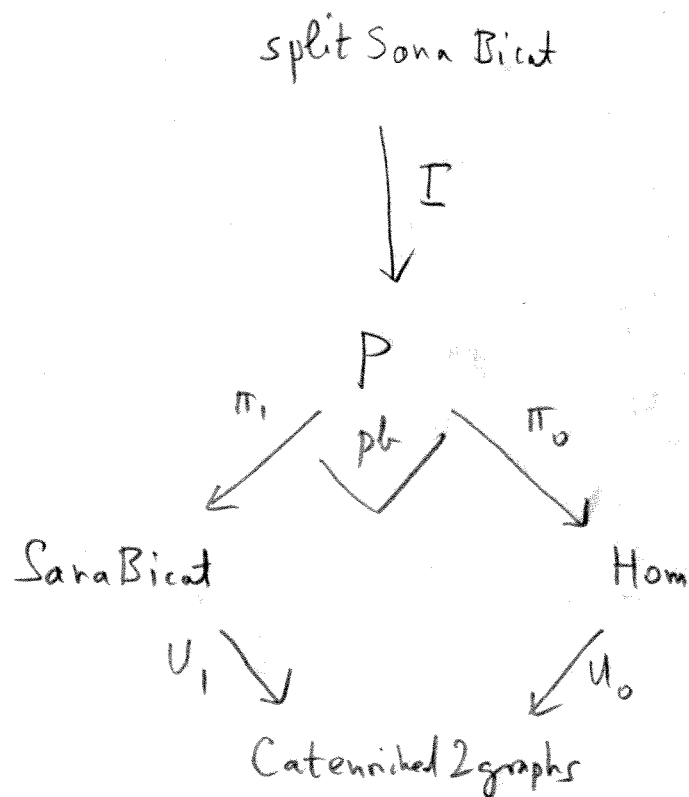
It is helpful to think of this expanded diagram when we deal with morphisms:

$$\begin{array}{ccccc}
 & & \downarrow & & \\
 & & (\tau, \bar{\sigma}, \bar{\varepsilon}, \theta) & & \\
 \swarrow \pi_0 & & & \searrow \pi_1 & \\
 (\tau, \bar{\sigma}, \theta) & & & & (\tau, \bar{\varepsilon}, \theta) \\
 \searrow U_1 & & & \swarrow U_0 & \\
 & (\tau, \theta) & & &
 \end{array}$$

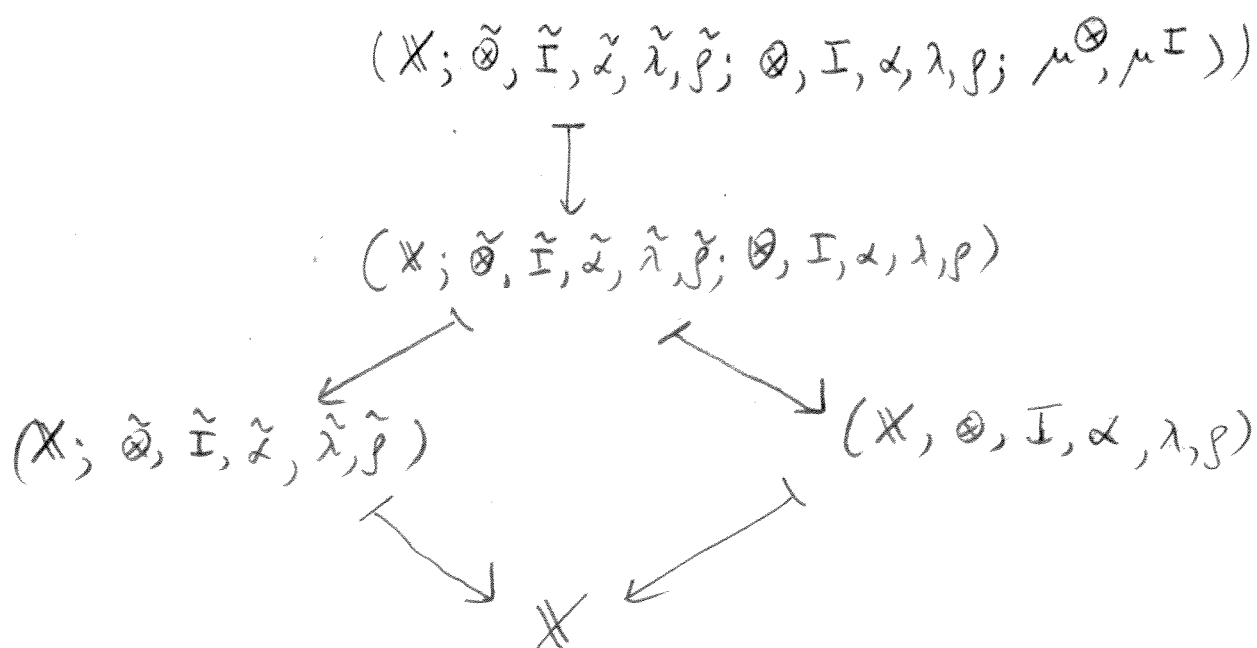
"($\tau, \bar{\sigma}, \bar{\varepsilon}, \theta$) with conditions "

(2.3)

There is going to be a similar situation in the present paper. We will have



Exemplified with objects (anticipating!):



In the last section (§ 5.4), I will give a formulation of the result that many will find preferable.

§2

Hom

Hom, of course, is classical — but, I need the notation going with it, so I give a full definition. (NB) In this, as well as the succeeding definitions, I find it very helpful to keep in mind the

(*) "stuff (elements)/operations/laws"

tripartite distinction. An (algebraic) structure such as a bicategory, is, first of all, a set of its elements within which we distinguish separate kinds (types). Secondly, there are operations — there are 'partial', or more precisely, conditional: they are defined for certain kinds of complexes (tuples) of elements, and give results that are of certain definite kinds. Finally, the laws are conditions, which, most frequently, are in the form of (conditional) identities, but, in cases that are significant for us, they may have (higher) logical complexity.

I learned about the 'tripartite division' of definitions from James Dolan; if I recall correctly, he talked about it in the Minneapolis

meeting on higher categories in 2004(?).

The point about the tripartite division is that the notion of morphism of structures defined in the so-articled manner will now have two aspects:

operations and laws.

(no elements other than those of the domain and the codomain), and, the operations-part of a morphism will refer only to the elements-part of the structures (domain and codomain), and the laws-part of the morphism will refer to the operations-part of the structures; the laws-part of the structure will have no role in the definition of "morphism"!

Let us see if our (numerous!) definitions are being made clearer by the above — even if we make no formal statement about the 'stuff' said above.

The elements and the operations together are referred to as data.

§2 The category Hom :

(5)

§2.1 Object of Hom : A tricategory - call it X - consists of :

(1) (rel) A category-enriched 2-graph X [nature of notation!] a set X_0 (the 0-cells of X), and, for each pair (X, Y) of elements of X_0 , a category $X(X, Y)$; (objects and arrows of $X(X, Y)$: 1-cells and 2-cells of X).

(2) (op & law) For any triple $(X, Y, Z) \in X_0^3$ of elements of X_0 , composition and identity

functors

$$(*) \quad \begin{cases} \otimes_{X, Y, Z} : X(X, Y) \times X(Y, Z) \longrightarrow X(X, Z) \\ I_X : \underline{1} \longrightarrow X(X, X) \end{cases}$$

terminal category

Abbreviations : for $U = (X, Y, Z)$, I write

$$\otimes_U : X(U) \longrightarrow \hat{X}(U)$$

for the line $(*)$

(3) (op & law) α, λ, ρ : the usual coherence

// natural isomorphisms - as follows :

3.1, 3.2, 3.3

(6)

(3.1) (op & law)

for any quadruple $(X, Y, Z, W) \in \mathbb{X}_0^4$,
 a natural isomorphism $\alpha^{X, Y, Z, W}$:

$$\alpha^{X, Y, Z, W} := \otimes_{X, Y, W} \circ (\mathbb{X}(X, Y) \times \otimes_{Y, Z, W})$$

$$\xrightarrow{\cong} \otimes_{X, Y, W} \circ (\otimes_{X, Y, Z} \times \mathbb{X}(Z, W));$$

or diagrammatically:

$$\begin{array}{ccc} \mathbb{X}(X, Y) \times \otimes_{Y, Z, W} & & \otimes_{X, Y, Z} \times \mathbb{X}(Z, W) \\ \downarrow & \xrightarrow{\alpha^{X, Y, Z, W}} & \downarrow \otimes_{X, Z, W} \\ \mathbb{X}(X, Y) \times \mathbb{X}(Y, W) & \xrightarrow{\otimes_{X, Y, W}} & \mathbb{X}(X, W) \end{array}$$

(The top right arrow is labeled with $\otimes_{X, Y, Z} \times \mathbb{X}(Z, W)$ above it, and the bottom right arrow is labeled with $\otimes_{X, Y, W}$ below it. There is also a $\cong$ symbol in a circle between the two top horizontal arrows.)

(NB) As usual, we write $X \xrightarrow{f} Y$, or $f: X \rightarrow Y$ for f an object of $\mathbb{X}(X, Y)$; and $\varphi: f \rightarrow g$, or $f \xrightarrow{\varphi} g$, for φ an arrow of $\mathbb{X}(X, Y)$; thus

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \downarrow \varphi & \\ & g & \end{array}$$

is a full notation for α and its dependencies.

We see that $\alpha = \alpha^{X,Y,Z,W}$ is an operation on

triples (f, g, h) of composable arrows (1-cells of $\mathbb{X}$)

$$\begin{array}{ccccc} & & g & & \\ & & \rightarrow & & \\ X & \xrightarrow{f} & Y & \xrightarrow{\quad} & Z & \xrightarrow{h} & W \end{array}$$

such that $\alpha(f, g, h)$, also written $\alpha_{f,g,h}$, is a 2-cell of $\mathbb{X}$

— an arrow in the category $\mathbb{X}(X, W)$ — of the following

kind:

$$\begin{array}{ccc} & \xrightarrow{\text{ho}(g \circ f)} & \\ X & \downarrow \alpha_{f,g,h} & W; \\ & \xrightarrow{\text{ho}(g \circ f)} & \end{array}$$

where we have used the (usual) abbreviation:

$$\text{so } \Gamma \stackrel{\text{def}}{=} \otimes_{A,B,C} (\Gamma, S)$$

for $A \xrightarrow{\Gamma} B \xrightarrow{S} C$ in $\mathbb{X}(A, B) \times \mathbb{X}(B, C)$,

four times. The facts that $\alpha^{X,Y,Z,W}$ should be

a natural transformation, and that its components be

isomorphism 2-cells are regarded law-requirements

('isomorphism' involves the existence of an inverse).

3.2) (op & law)

for any pair $(X, Y) \in \mathcal{X}_0^2$, a

natural isomorphism $\lambda^{X,Y}$:

$$\lambda^{X,Y}: \oplus_{X,Y,Y} \circ (\mathbb{X}(X,Y) \times I_Y) \xrightarrow{\cong} \text{Id}_{\mathbb{X}(X,Y)}$$

i.e.

$$\begin{array}{ccc} \mathbb{X}(X,Y) \times I_Y & \xrightarrow{\quad} & \mathbb{X}(X,Y) \times \mathbb{X}(Y,Y) \\ & \searrow \cong \downarrow \lambda^{X,Y} & \searrow \oplus_{X,Y,Y} \\ \mathbb{X}(X,Y)(\times 1) & \xrightarrow{\quad \text{Id}_{\mathbb{X}(X,Y)} \quad} & \mathbb{X}(X,Y) \end{array}$$

with components

$$\lambda_f : 1_Y \circ f \xrightarrow{\cong} f \quad (f: X \rightarrow Y, \text{ and}$$

$$1_f \stackrel{\text{def}}{=} I_Y(*) \in \mathbb{X}(Y,Y))$$

3.3) (op & law)

(9)

for any pair $(Y, Z) \in \mathcal{X}_0^2$, a
natural isomorphism $p^{Y, Z}$:

$$p^{Y, Z} : \bigotimes_{Y, Y, Z} \circ (I_Y \otimes \mathbb{X}(Y, Z)) \xrightarrow{\cong} \text{Id}_{\mathbb{X}(Y, Z)}$$

i.e.

$$\begin{array}{ccc} I_Y \otimes \mathbb{X}(Y, Z) & \xrightarrow{\quad} & \mathbb{X}(Y, Y) \times \mathbb{X}(Y, Z) \\ & \searrow \cong \downarrow p^{X, Y} & \swarrow \bigotimes_{Y, Y, Z} \\ (1 \times) \mathbb{X}(Y, Z) & \xrightarrow{\text{Id}_{\mathbb{X}(Y, Z)}} & \mathbb{X}(Y, Z) \end{array}$$

with components

$$p_g : \underbrace{g \circ 1_Y}_{(\text{can write: } g|_Y \dots)} \xrightarrow{\cong} g \quad (g: Y \rightarrow Z)$$

The above data are to satisfy the following
(additional) laws: (4):

(4.3) For any $X \in \mathbb{X}_0$:

$$\begin{array}{ccc}
 & X & \\
 \nearrow^{1_X} & & \nwarrow^{1_X} \\
 X & & X \\
 & \downarrow \lambda_{1_X} = \beta_{1_X} & \\
 & \text{---} & \\
 & 1_X &
 \end{array}$$

end of definition
of object of flow

In summary: the data for a bicategory look like

$$(\mathbb{X}, \otimes, \alpha, \lambda, \rho)$$

where: $\mathbb{X}$ is a category-enriched 2-graph;

$$\otimes = \langle \otimes_u \rangle_{u \in \mathbb{X}_0^3} \quad (\text{see p. (5)}),$$

$$\alpha = \langle \alpha^{X,Y,Z,W} \rangle_{(X,Y,Z,W) \in \mathbb{X}_0^4},$$

$$\lambda = \langle \lambda^{X,Y} \rangle_{(X,Y) \in \mathbb{X}_0^2},$$

$$\rho = \langle \rho^{Y,Z} \rangle_{(Y,Z) \in \mathbb{X}_0^2},$$

all described in detail above.

§ 2.2

Morphisms in $\underline{\text{Hom}}$:

Let X, X' be objects of $\underline{\text{Hom}}$. I am using previous notation, with primes for X' .
 A morphism $F: X \rightarrow X'$ consists of

1) a morphism

$$F: X \rightarrow X'$$

of the underlying category enriched 2-graphs; that is,

1.1) a function $F: X_0 \rightarrow X'_0$

and

1.2) for every $(X, Y) \in X_0^2$, a functor

$$F_{X,Y}: X(X, Y) \longrightarrow X'(FX, FY);$$

2) for every $u = (X, Y, Z) \in X_0^3$,

a natural isomorphism

$$\Sigma^u: \otimes'_{Fu} \circ F_u \xrightarrow{\sim} \hat{F}_u \circ \otimes_u$$

where we have used the following abbreviations:

with $U = (X, Y, Z)$,

$$X(U) = X(X, Y) \times X(Y, Z)$$

$$\hat{X}(U) = X(X, Z)$$

$$\otimes_U = \otimes_{X, Y, Z} : X(U) \longrightarrow \hat{X}(U)$$

$$FU = (FX, FY, FZ)$$

$$\otimes'_{FU} = \otimes'_{FX, FY, FZ} : X'(FU) \longrightarrow \hat{X}'(FU)$$

$$F_U = F_{X, Y} \times F_{Y, Z} : X(U) \longrightarrow X'(FU)$$

$$\hat{F}_U = F_{X, Z} : \hat{X}(U) \longrightarrow \hat{X}'(FU)$$

Diagram:

$$\begin{array}{ccc}
 X(U) & \xrightarrow{\otimes_U} & \hat{X}(U) \\
 F_U \downarrow & \xRightarrow{\Sigma^U} & \downarrow \hat{F}_U \\
 X'(FU) & \xrightarrow{\otimes'_{FU}} & \hat{X}'(FU)
 \end{array}$$

NB. The notation ' Σ ' is with reference to the similar notation used in "An application..." (see p. 1 there) except that the orientation is reversed. Since Σ^U is an isomorphism, this change is not of any consequence. See more on this later.

(3) for every $X \in \mathbb{X}_0$,

a natural isomorphism $\Sigma_X^X: F_{X,X} \circ I_X^X \xrightarrow{\sim} I_{FX}^{X'}$

$$\begin{array}{ccc}
 & I_X^X & \nearrow \mathbb{X}(X, X) \\
 1 & \cong \downarrow \Sigma_X^X & \downarrow F_{X,X} \\
 & I_{FX}^{X'} & \searrow \mathbb{X}'(FX, FX)
 \end{array}$$

In fact, Σ_X^X is a single 2-cell in $\mathbb{X}'$:

$$\Sigma_X^X: F(1_X) \xrightarrow{\sim} 1_{F(X)}.$$

The data in 1), 2), 3) are required

to satisfy the conditions 4), 5) and 6).

Note, for 4) below, that Σ^U ($U = (X, Y, Z)$)

has the following type of components:

with $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\Sigma_{f,g} = \Sigma_{f,g}^U: (Fg)(Ff) \xrightarrow{\sim} F(gf);$$

here, $Ff = F_{X,Y}(f)$; $(Fg)(Ff) = \otimes'_{FU}(Ff, Fg)$;
etc.

(4):

(4) For every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in X ,
the following commutes:

$$((Fh)(Fg)(Ff)) \xrightarrow{\alpha'_{Ff, Fg, Fh}} (Fh)((Fg)(Ff))$$

$$\begin{array}{c} (\Sigma_{g,h}^-) Ff \\ \downarrow \\ F(hg)(Ff) \end{array}$$

=

$$\begin{array}{c} (Fh) \Sigma_{f,g} \\ \downarrow \\ (Fh) F(gf) \end{array}$$

(*)

$$\begin{array}{c} \Sigma_{f,hg} \\ \downarrow \\ F((hg)f) \end{array}$$

$$\begin{array}{c} \downarrow \Sigma_{gf,h} \\ F(h(gf)) \end{array}$$

$$\xrightarrow{F(\alpha'_{f,g,h})}$$

Here, $\Sigma_{f,g}$ is $(\Sigma^U)_{f,g}$, with $U = (X, Y, Z)$;

similarly for $\Sigma_{g,h}$, etc.

The diagram is one of objects and arrows in the
category $X'(FX, FW)$.

(5) For $Y \xrightarrow{1_Y} Y \xrightarrow{g} Z$ in $\mathcal{X}$:

$$\begin{array}{ccc}
 FX & \xrightarrow{F(1_Y)} & FY \xrightarrow{Fg} FZ \\
 \downarrow F_Y & & \downarrow \Sigma_{FY, g} \\
 FX & \xrightarrow{1_{FX}} & FX \xrightarrow{Fg} FZ \\
 \downarrow \beta_{Fg} & & \downarrow F(\beta_g) \\
 & \searrow & \searrow \\
 & Fg & Fg
 \end{array} =
 \begin{array}{ccc}
 FY & \xrightarrow{F(1_Y)} & FY \xrightarrow{Fg} FZ \\
 \downarrow \Sigma_{FY, g} & & \downarrow F(\beta_g) \\
 FY & \xrightarrow{F(g|_Y)} & FY \xrightarrow{Fg} FZ \\
 \downarrow F(\beta_g) & & \downarrow F(\beta_g) \\
 & \searrow & \searrow \\
 & Fg & Fg
 \end{array}$$

i.e.,

$$\beta_{Fg} \circ ((Fg)(F_Y)) = F(\beta_g) \circ F_{1_Y, g}$$

(6) For $X \xrightarrow{f} Y \xrightarrow{1_Y} Y$ in $\mathcal{X}$; ...
 'same' for λ

End of definition
 of 'arrow' in How

(NB) The definition of a morphism $F: \mathbb{X} \rightarrow \mathbb{X}'$ in Hom is restated, with reference to "An application ...", as follows. $F = (F; \Sigma^{\otimes}, \Sigma^{\mathbb{I}}) = (F; (\Sigma^u)_{u \in \mathbb{X}_0^3}, (\Sigma^x)_{x \in \mathbb{X}_0})$

where we have:

1)* a morphism $F: \mathbb{X} \rightarrow \mathbb{X}'$ of the underlying cat-enriched 2-graphs;

2)* appropriate data Σ^u ($u \in \mathbb{X}_0^3$) such that we have morphisms in pFun:

$$(F_u, \Sigma^u, \hat{F}_u): (\mathbb{X}(u), \hat{\mathbb{X}}(u), \otimes_u^{\mathbb{X}}) \longrightarrow (\mathbb{X}'(Fu), \hat{\mathbb{X}}'(Fu), \otimes_{Fu}^{\mathbb{X}'})$$

3)* appropriate data Σ^x ($x \in \mathbb{X}_0$) such that we have morphisms in pFun:

$$(Id_1, \Sigma^x, F_{x,x}): (1, \mathbb{X}(x,x)) \longrightarrow (1, \mathbb{X}'(Fx, Fx));$$

these data are to satisfy conditions 4), 5) and 6) (unchanged).

Composition in $\mathcal{H}\text{ow}$: ^{use} composition of
morphisms of cat-enriched 2-graphs,
and (!) composition in $p\text{-}\underline{\text{Fun}}$.

however, one has to check that for:

$$X \xrightarrow{F} X' \xrightarrow{F'} X''$$

the composition

$$X \xrightarrow{F'F} X''$$

so defined satisfies conditions 4), 5) and 6).

I will draw the diagram for condition 4).

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. The diagram

on p (15) for $F'F$ in place of F is the

outside hexagon of the following (the six nodes of the hexagon are put in boxes): The diagram is a diagram of objects and arrows in the category

$X''(F'FX, F'FW)$.

(19)

$$\boxed{((F'F_h)(F'F_g))(F'F_f)}$$

$$\xrightarrow{\alpha_{F'F_f, F'F_g, F'h}^X}$$

$$\boxed{(F'F_h)((F'F_g)(F'F_f))}$$

(19)

$$F'_{F_g, F_h} F'_{F_f}$$



$$F'(F_h F_g)(F'F_f)$$

(1)

$$(F'F_h)(F'_{F_f, F_g})$$



$$(F'F_h)(F'(F_g F_f))$$

$$F'(F_{g,h}) F'_{F_f}$$



$$\boxed{(F'F(h_g))(F'F_f)}$$

$$\swarrow F'_{F_f, F_h F_g}$$

$$F'(\alpha_{F_f, F_g, F_h}^X)$$



$$\boxed{(F'F_h)(F'F(g_f))}$$

(3)

$$F'((F_h F_g)(F_f)) \xrightarrow{F'(\alpha_{F_f, F_g, F_h}^X)} F'((F_h)(F_g F_f))$$

$$F'_{F_f, F(h_g)}$$



$$F'(F(h_g)(F_f))$$

$$\swarrow F'(F_{g,h}, F_f)$$

$$F'(F_h F_{f,g})$$



$$F'((F_h)(F(g_f)))$$

(4)

$$F'(F_{f,h_g})$$



$$\boxed{F'F((h_g)f)}$$

$$\downarrow F'_{(g_f, h)}$$

$$\boxed{F'F(h(g_f))}$$

$$\xrightarrow{F'F(\alpha_{f,g,h}^X)}$$

The hexagon ① is the ' α -condition' (4), p.15)

for the morphism $F': X' \rightarrow X''$, instantiated at the triple $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{Fh} FW$ in X' .

The hexagon ④ is the α -condition for the morphism $F: X \rightarrow X'$, at (f, g, h) — with an application of the functor

$$F'_{FX, FW}: X'(FX, FW) \longrightarrow X''(F'FX, F'FW).$$

The quadrangle ② comes from the naturality condition for $(\Sigma^A)^{FX, FY, FW}$ in:

$$\begin{array}{ccc} X'(FX, FY) \times X'(FY, FW) & \xrightarrow{\otimes X'} & X'(FX, FW) \\ F' \downarrow & \xRightarrow{(\Sigma^A)^{FX, FY, FW}} & \downarrow F' \\ X''(F'FX, F'FY) \times X'(F'FY, F'FW) & \xrightarrow{\otimes X''} & X''(F'FX, F'FW) \end{array}$$

instantiated at the arrow

$$\begin{array}{ccc} (Ff, FhFg) & \longrightarrow & (Ff, F(hg)) \\ (1_{Ff}, F_{g,h}) & & \end{array} \quad \text{in the upper left}$$

corner. For ③: use $(\Sigma')^{FX, FZ, FW}$.

End of definition of
the category Flow.

§3

Sana Bicat

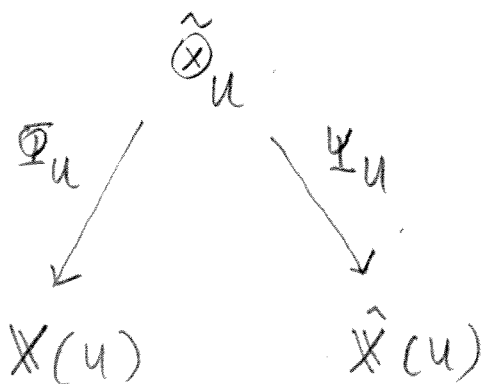
§3.1 Object of Sana Bicat: $\tilde{X}$: consist of:

1) $_{SB}$ A category-enriched 2-graph X — see above; X .

2) $_{SB}$ For any triple $u \in X_0^3$, a composition sana functor (object of Sana Fun: see "An application...")

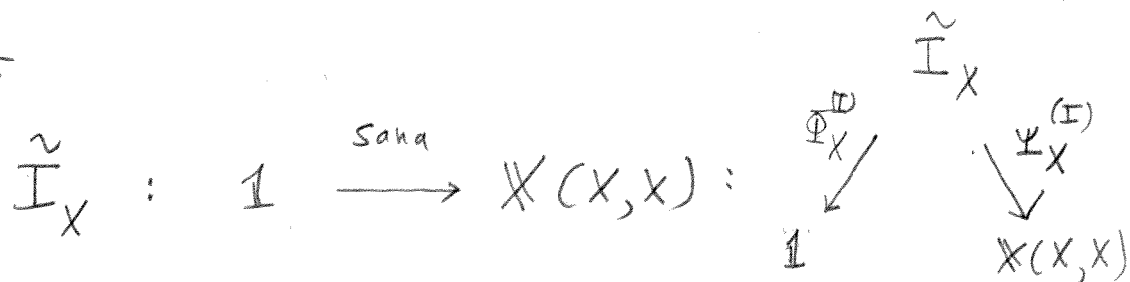
$$\tilde{\otimes}_u : \underbrace{X(u)}_{\text{see p ③}} \xrightarrow{\text{sana}} \hat{X}(u) ;$$

that is, a span of categories and functors



satisfying conditions given in "An application...".
and for each $X \in X_0$, an "identity"

sana functor



— To explain what α , λ and ρ are in

the sana-context, we use some abbreviated terminology.

Suppose the data in 1) & 2) are given.

Suppose $X, Y, Z \in X_0$, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ are in

$X(X, Y) \times X(Y, Z)$. Let $s \in \tilde{\otimes}_{X, Y, Z}$ — meaning that

s is an object of the category $\tilde{\otimes}_{X, Y, Z}$.

The expression $g \circ_s f$ (read: "the s -composite of f and g ")

is defined iff $\text{def}_{df}(f, g) = \Phi_{X, Y, Z}(s)$, and if so,

$g \circ_s f \stackrel{\text{def}}{=} \Psi_{X, Y, Z}(s)$. Notice that, since $\Phi_{X, Y, Z}$ is

Surjective on objects, for any $(f, g) \in X(X, Y) \times X(Y, Z)$, there is at least one $s \in \tilde{\otimes}_{X, Y, Z}$ for which $g \circ_s f$ is defined. Writing

$$\tilde{\otimes}_{X, Y, Z}(f, g)$$

for the fiber $\Phi_{X, Y, Z}^{-1}((f, g))$ of $\Phi_{X, Y, Z}$

$(\tilde{\otimes}_{X, Y, Z}(f, g))$: the subcategory of $\tilde{\otimes}_{X, Y, Z}$ with objects and arrows that map by $\Phi_{X, Y, Z}$ to the object (f, g) , resp. the identity arrow on (f, g) , of the category $X(U)$ ($U = (X, Y, Z)$), $g \circ_s f$ is defined iff $s \in \tilde{\otimes}_{X, Y, Z}(f, g)$.

Also note that if $g \circ_s f$ and $g' \circ_{s'} f'$ are both defined, then in fact $g = g'$ and $f = f'$: s determines f and g as well as $g \circ_s f$, in the expression $g \circ_s f$.

With the same bicat $\tilde{X}$, we have the further data (3.1), (3.2):

(3.1) SB: A family

$$\tilde{\alpha} = \langle \tilde{\alpha}^{X, Y, Z, W} \rangle_{(X, Y, Z, W) \in X_0^4}$$

and (3.3):

(23)

where, for a given quadruple (X, Y, Z, W) ,

$\tilde{\alpha}_{X,Y,Z,W}$ — simply $\tilde{\alpha}$ in what follows —

is an operation whose domain of definition, denoted $E = E^{X,Y,Z,W}$,

is the set of all quadruples (s_0, s_1, s_2, s_3) such that,

for suitable (uniquely determined) f, g, h as in

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$

each of: $g \circ_{s_0} f$, $h \circ_{s_1} g$, $(h \circ_{s_2} (g \circ_{s_0} f))$

and $(h \circ_{s_1} g) \circ_{s_3} f$

is defined, and the value at (s_0, s_1, s_2, s_3) ,

$\tilde{\alpha}_{s_0, s_1, s_2, s_3}$, is an arrow in $X(X, W)$ of the

following kind (an isomorphism):

$$\tilde{\alpha}_{s_0, s_1, s_2, s_3} : (h \circ_{s_1} g) \circ_{s_3} f \xrightarrow{\cong} h \circ_{s_2} (g \circ_{s_0} f).$$

Furthermore, we require that $\alpha_{s_0, s_1, s_2, s_3}$ be

natural in $\vec{s} \in (s_0, s_1, s_2, s_3)$ (it does not quite make sense to say: "natural in each of the variables s_0, s_1, s_2, s_3 ").

In fact, E is regarded as a full subcategory of

$$\tilde{\otimes}_{X,Y,Z} \times \tilde{\otimes}_{Y,Z,W} \times \tilde{\otimes}_{X,Z,W} \times \tilde{\otimes}_{X,Y,W}$$

and we have functors

$$E = E^{X,Y,Z,W} \begin{array}{c} \xrightarrow{K} \\ \xrightarrow{L} \end{array} X(X,W)$$

$$\text{for which } K(s_0, s_1, s_2, s_3) = (h_{s_1} g) \circ_{s_3} f$$

$$L(s_0, s_1, s_2, s_3) = h_{s_2} (g \circ_{s_0} f)$$

and the $\tilde{\otimes}_{X,Y,Z,W}$ datum is required to be a natural transformation, a natural isomorphism.

$$\tilde{\otimes}_{X,Y,Z,W} : K \xrightarrow{\cong} L.$$

Next, let us describe the items E, K, L more diagrammatically — this will ^{also} help verifying claim (**)

on p. ②, the fact that SanaBicat is dual-regular.

The composites of $\Phi_{A,B,C} : \tilde{\otimes}_{A,B,C} \longrightarrow X(A,B) \times X(B,C)$

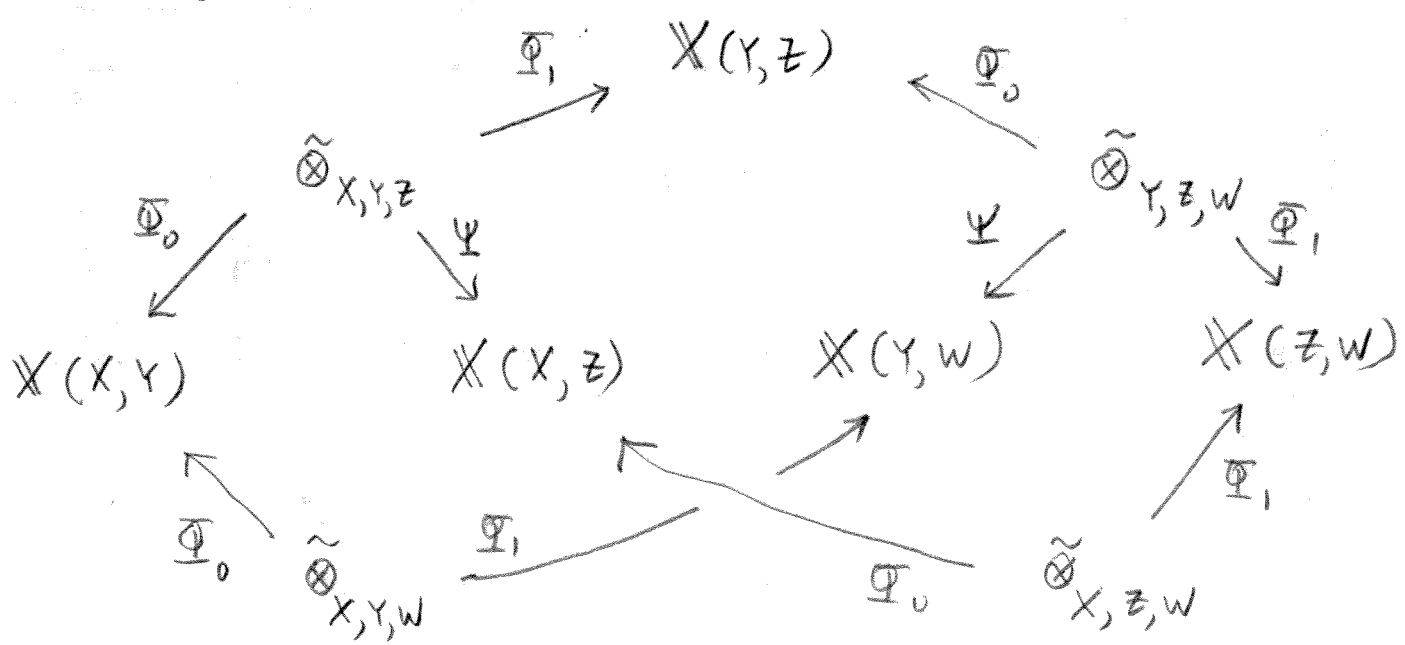
with product projections are denoted, with suppressing reference to

A, B, C , by

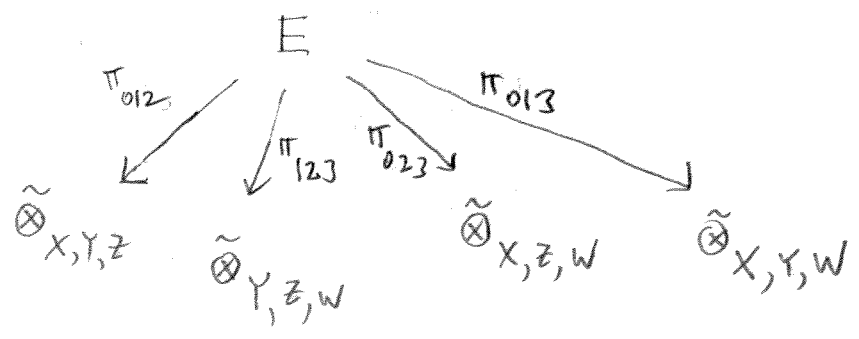
$$\Phi_0 : \tilde{\otimes}_{A,B,C} \longrightarrow X(A,B)$$

$$\Phi_1 : \tilde{\otimes}_{A,B,C} \longrightarrow X(B,C)$$

The category $E = E_{X,Y,Z,W}^{X,Y,Z,W}$ is defined as the limit (in Cat) of the following diagram of categories and functors:



The limit projections



determine the others as certain composites.

The functors

$$E \begin{matrix} \xrightarrow{K} \\ \xrightarrow{L} \end{matrix} X(X, W) \text{ are}$$

The composites in:

$$\begin{array}{ccccc}
 E^{X,Y,Z,W} & \xrightarrow{\pi_{013}} & \tilde{\otimes}_{X,Y,W} & \xrightarrow{\Psi_{X,Y,W}} & \\
 & & \cong \downarrow \alpha_{X,Y,Z,W} & & \\
 & & & & \text{X}(X,W) \\
 & \searrow \pi_{023} & \tilde{\otimes}_{X,Z,W} & \xrightarrow{\Psi_{X,Z,W}} & \\
 & & & &
 \end{array}$$

as indicated, $\alpha_{X,Y,Z,W}$ is, by definition, a natural isomorphism between the composite functors.

(NR) Compare the above to this:

$$\begin{array}{ccccccc}
 & & \Phi_1 & \rightarrow & g & \leftarrow & \Phi_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \Phi_0 & s_0 & \xrightarrow{\Psi} & g \circ s_0 f & \xrightarrow{\Psi} & h \circ s_1 g \\
 & \downarrow \Phi_0 & \downarrow \Phi_0 & & \downarrow \Phi_0 & & \downarrow \Phi_1 \\
 f & & s_3 & \xrightarrow{\Psi} & (h \circ s_1 g) \circ s_3 f & \xrightarrow{\Psi} & h \\
 & & \downarrow \Phi_1 & & \downarrow \Phi_1 & & \downarrow \Phi_1 \\
 & & s_2 & \xrightarrow{\Psi} & h \circ s_2 (g \circ s_0 f) & &
 \end{array}$$

$\alpha_{s_0, s_1, s_2, s_3}$

(3.2)
SB

A family

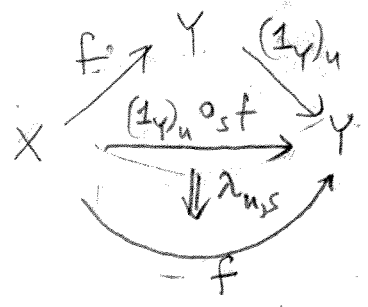
$$\lambda = \langle \lambda^{X,Y} \rangle_{(X,Y) \in \mathbb{X}_0^2}$$

such that $\lambda = \lambda^{X,Y}$ is a mapping with domain of definition the set of pairs (u, s)

such that $u \in \tilde{I}_X$, and, with $(1_Y)_u \stackrel{\text{def}}{=} \Psi_Y^{(I)}(u)$, we have $s \in \tilde{\otimes}_{X,Y,Y}(f, (1_Y)_u)$, and its value,

denoted $\lambda_{u,s}$ is an arrow: $\lambda_{u,s} : (1_Y)_u \circ s \xrightarrow{\sim} f$

in $\mathbb{X}(X, Y)$:

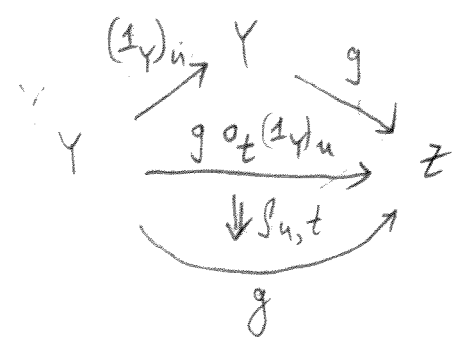


It is required that $\lambda_{u,s}$ be natural in (u, s) .

(3.3)
SB

(abbreviated)

$$\beta = \langle \beta^{Y,Z} \rangle_{(Y,Z) \in \mathbb{X}_0^2}$$



$\beta_{u,t}$: natural in (u, t) .

I need some more notation to state the conditions the above data are to satisfy.

Assuming we have the above, let $A, B, C \in \mathcal{X}_0$ and consider the span

$$\begin{array}{ccc} & \tilde{\otimes}_{A,B,C} & \\ \bar{\Phi}_{A,B,C} \swarrow & & \searrow \Psi_{A,B,C} \\ \mathcal{X}(A,B) \times \mathcal{X}(B,C) & & \mathcal{X}(A,C) \end{array}$$

Let $\mu: f \rightarrow f'$ be an arrow in $\mathcal{X}(A,B)$,

$\nu: g \rightarrow g'$ one in $\mathcal{X}(B,C)$. Let $s \in \tilde{\otimes}_{A,B,C}(f,g)$,

$s' \in \tilde{\otimes}_{A,B,C}(f',g')$ so that $g \circ_s f$ and $g' \circ_{s'} f'$ are defined ($g \circ_s f = \Psi_{A,B,C}(s)$, $g' \circ_{s'} f' = \Psi_{A,B,C}(s')$).

Since $\bar{\Phi}_{A,B,C}$ is full and faithful, there is a unique

$\beta_{\mu,\nu}^{s,s'} = \beta: s \rightarrow s'$ such that $\bar{\Phi}_{A,B,C}(\beta) = (\mu, \nu)$, the arrow $(\mu, \nu): (f,g) \rightarrow (f',g')$ in $\mathcal{X}(A,B) \times \mathcal{X}(B,C)$.

We denote the arrow in $\mathcal{X}(A,C)$:

$$\Psi_{A,B,C}(\beta): g \circ_s f \longrightarrow g' \circ_{s'} f'$$

$$\text{by: } \nu \circ_{s,s'} \mu: g \circ_s f \longrightarrow g' \circ_{s'} f'.$$

In case $\mu = 1_f : f \rightarrow f'$, I write $\nu \circ_{s,s',f}$
 for $\nu \circ_{s,s',\mu}$; and similarly when ν is an identity.

Before getting to ^{the} (familiar-looking) "Mac Lane conditions",
 let me express the naturality of $\tilde{\alpha}$ (see: p. 27) in the just-
 introduced notation:

Given

$$\left. \begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & W \\ & \downarrow \varphi & & \downarrow \gamma & & \downarrow \eta & \\ & \hat{f} & & \hat{g} & & \hat{h} & \end{array} \right\} (*)$$

and $s_0, s_1, s_2, s_3; \hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3$

such that all the objects in the following diagram
 are defined, we have that the diagram commutes:

$$\begin{array}{ccc} (h \circ_{s_1} g) \circ_{s_3} f & \xrightarrow{\tilde{\alpha}_{s_0, s_1, s_2, s_3}} & h \circ_{s_2} (g \circ_{s_0} f) \\ \downarrow & \text{=} & \downarrow \\ (\eta \circ_{s_1, \hat{s}_1} \gamma) \circ_{s_3, \hat{s}_3} \varphi & & \eta \circ_{s_2, \hat{s}_2} (\gamma \circ_{s_0, \hat{s}_0} \varphi) \\ & & \downarrow \\ (\hat{h} \circ_{\hat{s}_1} \hat{g}) \circ_{\hat{s}_3} \hat{f} & \xrightarrow{\tilde{\alpha}_{\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3}} & \hat{h} \circ_{\hat{s}_2} (\hat{g} \circ_{\hat{s}_0} \hat{f}) \end{array}$$

4.2)

SB

(abbreviated):

$$X \xrightarrow{f} Y \xrightarrow{(1_Y)_u} Y \xrightarrow{g} Z$$

$$(g \circ_{s_1} (1_Y)_u) \circ_{s_3} f \xrightarrow{\alpha_{s_0, s_1, s_2, s_3}} g \circ_{s_2} ((1_Y)_u \circ_s f)$$

$$\begin{array}{ccc} & \searrow & \swarrow \\ \rho_{u, s_1} \circ_{s_3, s_4} f & & g \circ_{s_2, s_4} \lambda_{u, s_0} \\ & \searrow & \swarrow \\ & g \circ_{s_4} f & \end{array} \quad (=)$$

4.3)

SB

(abbreviated):

$$\begin{array}{ccc} & (1_X)_v & X \\ & \nearrow & \searrow \\ X & & X \\ & (1_X)_v \circ_s (1_X)_v & \\ & \xrightarrow{\quad} & \\ & \lambda_{v, s} \downarrow = \downarrow \rho_{v, s} & \\ & (1_X)_v & \end{array}$$

$$\boxed{\lambda_{v, s} = \rho_{v, s}}$$

end of definition of a
sana-bicat: object of SanaBicat

§ 3.2 Morphism in Sana Bicat

Morphism of sana bicategories:

$$(F, \sigma^{\otimes}, \sigma^I) : (X, \tilde{\otimes}, \tilde{I}, \alpha, \lambda, \rho)$$

$$\longrightarrow (X', \tilde{\otimes}', \tilde{I}', \alpha', \lambda', \rho')$$

consists of:

(1) $F: X \longrightarrow X'$; morphism of category-enriched 2-graphs (as in Hony);

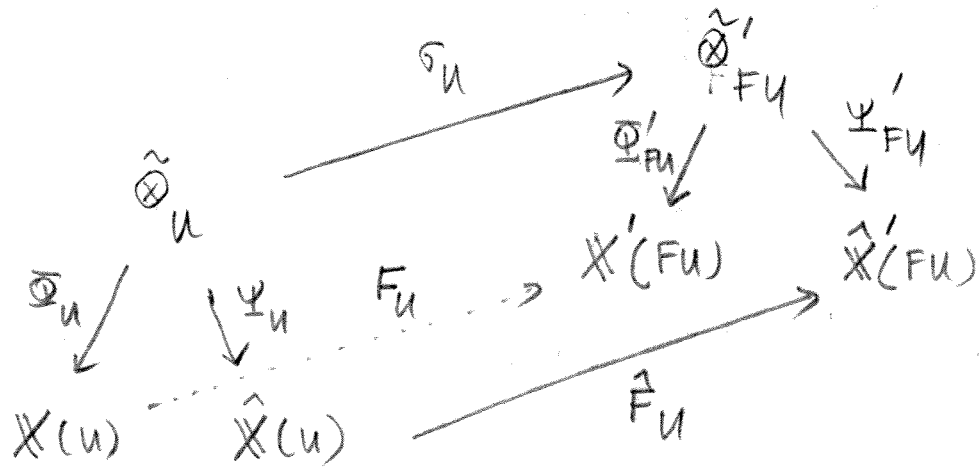
(2) $\sigma^{\otimes} = \langle \sigma_u \rangle_{u \in X_0}$, where each σ_u is a

a functor $\sigma_u: \tilde{\otimes}_u \longrightarrow \tilde{\otimes}'_{Fu}$

such that σ_u , together with F_u and $\hat{F}_u$, make up a morphism

$$(F_u, \sigma_u, \hat{F}_u): \tilde{\otimes}_u \longrightarrow \tilde{\otimes}'_{Fu}$$

of sana functors;



i.e., a (strict) morphism η spans:

$$F_u \circ \Phi_u = \Phi'_{Fu} \circ \sigma_u$$

$$\hat{F}_u \circ \Psi_u = \Psi'_{Fu} \circ \sigma_u$$

(See: "An application ...").

$$(3) \quad \sigma^I = \langle \sigma_X^I \rangle_{X \in X_0} :$$

$$(!, \sigma_X^I, F_{X,X}) : \tilde{I}_X \longrightarrow \tilde{I}_{FX}$$

a morphism in SanaFun.

The above data are required to preserve α, λ, ρ :

$$(4) \quad \text{For } \tilde{\alpha}: \quad \text{for } X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

$$\text{and } \tilde{\alpha}_{s_0, s_1, s_2, s_3} : (h \circ_{s_1} g) \circ_{s_2} f \longrightarrow h \circ_{s_2} (g \circ_{s_0} f)$$

in the category $X(X, W)$, I abbreviate

$$s_0' = \sigma_{X,Y,Z}(s_0)$$

$$s_1' = \sigma_{Y,Z,W}(s_1)$$

$$s_2' = \sigma_{X,Z,W}(s_2)$$

$$s_3' = \sigma_{X,Y,W}(s_3) ;$$

by what we already know, we have that

$$F((h_{o_{s_1}g})_{o_{s_3}}f) = (Fh_{o_{s_1}}Fg)_{o_{s_3'}}Ff$$

(where, of course, Ff abbreviates $F_{X,Y}(f)$, etc.),

and similarly for the other triple composite;

the new requirement is

$$(*) \quad F_{X,W}(\alpha_{s_0,s_1,s_2,s_3}) = \alpha'_{s_0',s_1',s_2',s_3'}$$

(5), 6) Entirely similar preservation requirements are in place for λ and ρ .

end of definition
of morphism in
Sara bicat's and of
the category SaraBicat