Applications of anafunctors II
M. Makkai / Nov 12, 014

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Applications of ana-functor II.

M. Makkai

§1 With Ross Street (Cahiers TCDC, 1980), we denote by Hom the category (!) of small bicategories and homomorphisms (= strong morphisms) ("small" is relative to an arbitrary choice of a Grothendieck universe)

Theorem Hom is a dual-regular category.

NB. For 'dual-regular', see "An application of anafunctors (July 7, 2014)", p. 3 (ref. [1])

I repeat the definition:

Definition Category $\mathbb{A}$ is dual-regular if there exists a small regular category R such that

$$\mathbb{A} \underset{\substack{\uparrow \\ \text{category equivalence}}}{\simeq} \mathrm{Reg}(R, \mathrm{Set})\ ;$$

here, $\mathrm{Reg}(R, \mathrm{Set})$ is the category of all regular functors $R \to \mathrm{Set}$; $\mathrm{Reg}(R, \mathrm{Set})$ is a full subcategory of $[R, \mathrm{Set}]$.

NBNB In the definition, the word 'small' is coordinated with the definition of Set: the latter is the category of <u>small</u> sets. Of the category $\mathbb{A}$, the initial assumption is merely that it be <u>locally</u> <u>small</u>: all hom-sets $\mathbb{A}(A, B)$ are to be <u>small</u>.

The proof of the theorem will be given by constructing two further categories and (forgetful) functors shown next:

$$
\begin{array}{ccc}
 & \underline{\text{split}}\ \underline{\text{Sana Bicat}} = \underline{\text{sSB}} & \\
\swarrow S & & R \searrow \\
\underline{SB} = \underline{\text{Sana Bicat}} & & \underline{\text{Hom}};
\end{array}
$$

and proving that:

(*) R and S are both full and faithful and surjective on objects;

and pointing out the essentially obvious fact that

(**) <u>Sana Bicat</u> is dual-regular.

Further preliminary remarks:

The present write-up continues and applies "An application of anafunctors" (July 7, 2014) by the same author.

In that paper underlying the work there, we have the span:

$$\begin{tikzcd}
 & \mathrm{SanapFun} \arrow[dl, "S"'] \arrow[dr, "R"] & \\
\mathrm{Sana}\underbrace{\mathrm{Fun}}_{\text{added now}} & & \mathrm{pFun}
\end{tikzcd}$$

see p. 6, loc. cit. Let me clarify the construction there, and also here, by expanding the spans into bigger diagrams: we have, in the previous papers:

$$\begin{tikzcd}
 & \mathrm{SanapFun} \arrow[d, "I"] \arrow[ddl, bend right, "S"'] \arrow[ddr, bend left, "R"] & \\
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] \arrow[l, phantom, "="] \arrow[r, phantom, "="] \arrow[d, phantom, "\mathrm{pb}"] & \\
\mathrm{Sana\,Fun} \arrow[dr, "U_1"'] & & \mathrm{p\,Fun} \arrow[dl, "U_0"] \\
 & \mathrm{Cat}\times\mathrm{Cat} &
\end{tikzcd}$$

P here is the pullback of the two forgetful functors U_0, U_1:

$$(\Phi, X, A) \xmapsto{\ U_0\ } (X, A)$$

$$(|F|, X, A, \varphi, \psi) \xmapsto{\ U_1\ } (X, A)$$

(with references to the notation in "An application ...")

and I the forgetful functor

$$(|F|, X, A, \bar{\Phi}, \varphi, \psi, \mu) \xmapsto{\ I\ } (|F|, X, A, \Phi, \varphi, \psi)$$

"forget μ"

It is helpful to think of this expanded diagram when we deal with morphisms:

$$\begin{tikzcd}
 & \text{"}(\tau, \sigma, \Sigma, \theta) \text{ with conditions"} \arrow[d, mapsto, "I"] & \\
 & (\tau, \sigma, \Sigma, \theta) \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\
(\tau, \sigma, \theta) \arrow[dr, mapsto, "U_1"'] & & (\tau, \Sigma, \theta) \arrow[dl, mapsto, "U_0"] \\
 & (\tau, \theta) &
\end{tikzcd}$$

There is going to be a similar situation in the present paper. We will have

$$\begin{tikzcd}
 & \text{split SaraBicat} \arrow[d, "I"] & \\
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] \arrow[d, phantom, "pb"] & \\
\text{SaraBicat} \arrow[dr, "U_1"'] & & \text{Hom} \arrow[dl, "U_0"] \\
 & \text{Catenriched 2graphs} &
\end{tikzcd}$$

Exemplified with objects (anticipating!):

$$\begin{tikzcd}
 & (\mathbb{X};\tilde{\otimes},\tilde{I},\tilde{\alpha},\tilde{\lambda},\tilde{\rho};\otimes,I,\alpha,\lambda,\rho;\mu^{\otimes},\mu^{I}) \arrow[d, mapsto] & \\
 & (\mathbb{X};\tilde{\otimes},\tilde{I},\tilde{\alpha},\tilde{\lambda},\tilde{\rho};\otimes,I,\alpha,\lambda,\rho) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(\mathbb{X};\tilde{\otimes},\tilde{I},\tilde{\alpha},\tilde{\lambda},\tilde{\rho}) \arrow[dr, mapsto] & & (\mathbb{X},\otimes,I,\alpha,\lambda,\rho) \arrow[dl, mapsto] \\
 & \mathbb{X} &
\end{tikzcd}$$

In the last section (§ 5.4), I will give a formulation of the result that many will find preferable.

§2 Hom

Hom, of course, is classical – but, I need the notation going with it, so I give a full definition. (NB) In this, as well as the succeeding definitions, I find it very helpful to keep in mind the

(*) "stuff (elements) / operations / laws"

tripartite distinction. An (algebraic) structure such as a bicategory, is, first of all, a set of its elements within which we distinguish separate kinds (types). Secondly, there are operations – these are 'partial', or more precisely, conditional: they are defined for certain kinds of complexes (tuples) of elements, and give results that are of certain definite kinds. Finally, the laws are conditions, which, most frequently, are in the form of (conditional) identities, but, in cases that are significant for us, they may have (higher) logical complexity.

I learned about the 'tripartite division' of definitions from James Dolan; if I recall correctly, he talked about it in the Minneapolis meeting on higher categories in 2004(?).

The point about the tripartite division is that the notion of morphism of structures defined in the so-articuled manner will now have two aspects:

operations and laws.

(no elements other than those of the domain and the codomain), and, the operations-part of a morphism will refer only to the elements-part of the structures (domain and codomain), and the laws-part of the morphism will refer to the operations-part of the structures; the laws-part of the structure will have no role in the definition of "morphism"!

Let us see if our (numerous!) definitions are being made clearer by the above — even if we make no formal statement about the 'stuff' said above.

The elements and the operations together are referred to as data.

§2 The category Hom :

§2.1 Objects of Hom :

A bicategory – call it $\mathbb{X}$ – consists of:

1) (el) A category-enriched 2-graph $\mathbb{X}$ [abuse of notation!]

a set $\mathbb{X}_0$ (the 0-cells of $\mathbb{X}$), and, for each pair (X,Y) of elements of $\mathbb{X}_0$, a category $\mathbb{X}(X,Y)$; (objects and arrows of $\mathbb{X}(X,Y)$: 1-cells and 2-cells of $\mathbb{X}$).

2) (op & law) For any triple $(X,Y,Z)\in\mathbb{X}_0^3$ of elements of $\mathbb{X}_0$, composition and identity functors

$$(*)\quad \begin{cases} \otimes_{X,Y,Z} : \mathbb{X}(X,Y)\times\mathbb{X}(Y,Z) \longrightarrow \mathbb{X}(X,Z) \\ I_X : \underbrace{\mathbb{1}}_{\text{terminal category}} \longrightarrow \mathbb{X}(X,X) \end{cases}$$

Abbreviations: for $\mathcal{U}=(X,Y,Z)$, I write

$$\otimes_{\mathcal{U}} : \mathbb{X}(\mathcal{U}) \longrightarrow \hat{\mathbb{X}}(\mathcal{U})$$

for the line $(*)$

3) (op & law) α,λ,ρ : the usual coherence natural isomorphisms – as follows :

//

3.1, 3.2, 3.3

(3.1) (op-law)

for any quadruple $(X,Y,Z,W) \in \mathbb{X}_0^4$,

a natural isomorphism $\alpha^{X,Y,Z,W}$:

$$\alpha^{X,Y,Z,W} : \otimes_{X,Y,W} \circ (\mathbb{X}(X,Y) \times \otimes_{Y,Z,W})$$

$$\xrightarrow{\cong} \otimes_{X,Y,W} \circ (\otimes_{X,Y,Z} \times \mathbb{X}(Z,W)) \; ;$$

or diagrammatically:

$$\begin{tikzcd}
\mathbb{X}(X,Y) \times \mathbb{X}(Y,Z) \times \mathbb{X}(Z,W) \arrow[r, "\otimes_{X,Y,Z} \times \mathbb{X}(Z,W)"] \arrow[d, "\mathbb{X}(X,Y) \times \otimes_{Y,Z,W}"'] \arrow[dr, phantom, "\overset{\alpha^{X,Y,Z,W}}{\Rightarrow} \cong"] & \mathbb{X}(X,Z) \times \mathbb{X}(Z,W) \arrow[d, "\otimes_{X,Z,W}"] \\
\mathbb{X}(X,Y) \times \mathbb{X}(Y,W) \arrow[r, "\otimes_{X,Y,W}"'] & \mathbb{X}(X,W) .
\end{tikzcd}$$

NB As usual, we write $X \xrightarrow{f} Y$, or $f: X \to Y$ for f an object of $\mathbb{X}(X,Y)$; and $\varphi: f \to g$, or $f \xrightarrow{\varphi} g$, for φ an arrow of $\mathbb{X}(X,Y)$; thus

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varphi"] & Y
\end{tikzcd}$$

is a full notation for φ and its dependencies.
We see that $\alpha = \alpha^{X,Y,Z,W}$ is an <u>operation</u> on triples (f, g, h) of <u>composable</u> <u>arrows</u> (<u>1-cells</u> of $\mathbb{X}$)

$$\begin{tikzcd} & Y \arrow[r, "g"] & Z \arrow[dr, "h"] & \\ X \arrow[ur, "f"] & & & W \end{tikzcd}$$

such that $\alpha((f,g,h))$, also written $\alpha_{f,g,h}$, is a <u>2-cell</u> of $\mathbb{X}$ – an arrow in the category $\mathbb{X}(X, W)$ – of the following kind:

$$\begin{tikzcd} X \arrow[r, bend left, "(h\circ g)\circ f"] \arrow[r, phantom, "\Downarrow\alpha_{f,g,h}"] \arrow[r, bend right, "h\circ(g\circ f)"'] & W \end{tikzcd};$$

<u>where</u> we have used the (usual) abbreviation:

$$s \circ r \stackrel{\text{def}}{=} \otimes_{A,B,C}(r, s)$$

for $A \xrightarrow{r} B \xrightarrow{s} C$ in $\mathbb{X}(A,B) \times \mathbb{X}(B,C)$, <u>four</u> times. The facts that $\alpha^{X,Y,Z,W}$ should be a <u>natural</u> transformation, and that its components be isomorphism 2-cells are regarded <u>law</u>-requirements ('isomorphism' involves the <u>existence</u> of an inverse).

(3.2) (op. f law)

for any pair $(X,Y) \in \mathbb{X}_0^2$, a

natural isomorphism $\lambda^{X,Y}$:

$$\lambda^{X,Y} : \otimes_{X,Y,Y} \circ (\mathbb{X}(X,Y) \times I_Y) \xrightarrow{\cong} \mathrm{Id}_{\mathbb{X}(X,Y)}$$

i.e.

$$\begin{tikzcd}
 & \mathbb{X}(X,Y) \times \mathbb{X}(Y,Y) \arrow[dr, "\otimes_{X,Y,Y}"] \arrow[d, phantom, "\cong \Downarrow \lambda^{X,Y}"] & \\
\mathbb{X}(X,Y)(\times \mathbb{1}) \arrow[ur, "\mathbb{X}(X,Y) \times I_Y"] \arrow[rr, "\mathrm{Id}_{\mathbb{X}(X,Y)}"'] & & \mathbb{X}(X,Y)
\end{tikzcd}$$

with components

$$\lambda_f : 1_Y \circ f \xrightarrow{\cong} f \quad (f : X \to Y, \text{ and}$$

$$1_f \underset{\text{def}}{=} I_Y(*) \ (\in \mathbb{X}(Y,Y))$$

3.3) (op. & law)

for any pair $(Y, Z) \in \mathbb{X}_0^2$, a

natural isomorphism $\rho^{Y,Z}$:

$$\rho^{Y,Z} : \otimes_{Y,Y,Z} \circ (I_Y \otimes \mathbb{X}(Y,Z)) \xrightarrow{\cong} \mathrm{Id}_{\mathbb{X}(Y,Z)}$$

i.e.

$$\begin{tikzcd}
 & \mathbb{X}(Y,Y) \times \mathbb{X}(Y,Z) \arrow[d, phantom, "\cong\,\Downarrow \rho^{Y,Y}"] \arrow[dr, "\otimes_{Y,Y,Z}"] & \\
(\mathbb{1} \times) \mathbb{X}(Y,Z) \arrow[ur, "I_Y \otimes \mathbb{X}(Y,Z)"] \arrow[rr, "\mathrm{Id}_{\mathbb{X}(Y,Z)}"'] & & \mathbb{X}(Y,Z)
\end{tikzcd}$$

with components

$$\rho_g : \underbrace{g \circ 1_Y}_{\text{(can write: } g 1_Y \ldots)} \xrightarrow{\cong} g \qquad (g : Y \to Z)$$

The above data are to satisfy the following (additional) laws: (4):

4) (no surprise!)

4.1) (Mac Lane's pentagon)

Given

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & W \arrow[r, "i"] & V \end{tikzcd}$$

in $\mathbb{X}$, the following commutes ($gf = g \circ f$, etc.):

$$\begin{tikzcd}
 & (ih)(gf) \arrow[dr, "\alpha_{gf,h,i}"] & \\
((ih)g)f \arrow[ur, "\alpha_{f,g,ih}"] \arrow[d, "\alpha_{g,h,i}f"'] \arrow[rr, phantom, "="] & & i(h(gf)) \\
(i(hg))f \arrow[r, "\alpha_{f,hg,i}"'] & i((hg)f) \arrow[ur, "i\alpha_{f,g,h}"'] &
\end{tikzcd}$$

4.2) (Identity coherence)

Following the notation in 3.2) and 3.3):

$$\begin{tikzcd}
(g1_Y)f \arrow[rr, "\alpha_{f,1_Y,g}"] \arrow[dr, "\rho_g f"'] \arrow[rr, phantom, shift right=4, "="] & & g(1_Y f) \arrow[dl, "g\lambda_f"] \\
 & gf &
\end{tikzcd}$$

4.3) For any $X \in \mathbb{X}_0$:

$$\begin{tikzcd} & X \arrow[dr, "1_X"] & \\ X \arrow[ur, "1_X"] \arrow[rr, bend right, "1_X"'] & \Downarrow \lambda_{1_X} = \rho_{1_X} & X \end{tikzcd}$$

$\boxed{\text{end of definition of object of Hom}}$

In summary: the data for a bicategory look like

$$(\mathbb{X}, \otimes, \alpha, \lambda, \rho)$$

where: $\mathbb{X}$ is a category-enriched 2-graph;

$\otimes = \langle \otimes_U \rangle_{U \in \mathbb{X}_0^3}$ (see p. 5),

$\alpha = \langle \alpha^{X,Y,Z,W} \rangle_{(X,Y,Z,W) \in \mathbb{X}_0^4}$,

$\lambda = \langle \lambda^{X,Y} \rangle_{(X,Y) \in \mathbb{X}_0^2}$,

$\rho = \langle \rho^{Y,Z} \rangle_{(Y,Z) \in \mathbb{X}_0^2}$,

all described in detail above.

§2.2 Morphisms in $\underline{\mathrm{Hom}}$:

Let $\mathbb{X}$, $\mathbb{X}'$ be objects of $\underline{\mathrm{Hom}}$. I am using previous notation, with primes for $\mathbb{X}'$;
A morphism $F: \mathbb{X} \longrightarrow \mathbb{X}'$ consists of

1) a morphism

$$F: \mathbb{X} \longrightarrow \mathbb{X}'$$

of the underlying category enriched 2-graphs; that is,

1.1) a function $F: \mathbb{X}_0 \longrightarrow \mathbb{X}'_0$

and

1.2) for every $(X,Y) \in \mathbb{X}_0^{\,2}$, a functor

$$F_{X,Y}: \mathbb{X}(X,Y) \longrightarrow \mathbb{X}'(FX,FY);$$

2) for every $\mathcal{U} = (X,Y,Z) \in \mathbb{X}_0^{\,3}$,

a natural isomorphism

$$\Sigma^{\mathcal{U}}: \otimes'_{F\mathcal{U}} \circ F_{\mathcal{U}} \xrightarrow{\ \cong\ } \hat{F}_{\mathcal{U}} \circ \otimes_{\mathcal{U}}$$

where we have used the following abbreviations:

with $\mathcal{U} = (X, Y, Z)$,

$$\mathbb{X}(\mathcal{U}) = \mathbb{X}(X,Y) \times \mathbb{X}(Y,Z)$$

$$\hat{\mathbb{X}}(\mathcal{U}) = \mathbb{X}(X,Z)$$

$$\otimes_{\mathcal{U}} = \otimes_{X,Y,Z} \quad : \mathbb{X}(\mathcal{U}) \longrightarrow \hat{\mathbb{X}}(\mathcal{U})$$

$$F\mathcal{U} = (FX, FY, FZ)$$

$$\otimes'_{F\mathcal{U}} = \otimes'_{FX,FY,FZ} \quad : \mathbb{X}'(F\mathcal{U}) \longrightarrow \hat{\mathbb{X}}'(F\mathcal{U})$$

$$F_{\mathcal{U}} = F_{X,Y} \times F_{Y,Z} : \mathbb{X}(\mathcal{U}) \longrightarrow \mathbb{X}'(F\mathcal{U})$$

$$\hat{F}_{\mathcal{U}} = F_{X,Z} \quad : \hat{\mathbb{X}}(\mathcal{U}) \longrightarrow \hat{\mathbb{X}}'(F\mathcal{U}).$$

Diagram:

$$\begin{tikzcd}
\mathbb{X}(\mathcal{U}) \arrow[r, "\otimes^{\mathbb{X}}_{\mathcal{U}}"] \arrow[d, "F_{\mathcal{U}}"'] \arrow[dr, phantom, "\overset{\Sigma^{\mathcal{U}}}{\underset{\cong}{\Rightarrow}}"] & \hat{\mathbb{X}}(\mathcal{U}) \arrow[d, "\hat{F}_{\mathcal{U}}"] \\
\mathbb{X}'(F\mathcal{U}) \arrow[r, "\otimes^{\mathbb{X}'}_{F\mathcal{U}}"'] & \hat{\mathbb{X}}'(F\mathcal{U})
\end{tikzcd}$$

N.B. The notation 'Σ' is with reference to the similar notation used in "An application..." (see p. 1 there) except that the orientation is reversed. Since $\Sigma^{\mathcal{U}}$ is an isomorphism, this change is not of any consequence. See more on this later.

3) for every $X \in \mathbb{X}_0$,

a natural isomorphism $\Sigma^X : F_{X,X} \circ I_X^{\mathbb{X}} \xrightarrow{\cong} I_{FX}^{\mathbb{X}'}$:

$$\begin{tikzcd}
 & \mathbb{X}(X,X) \arrow[d, "F_{X,X}"] \\
\mathbb{1} \arrow[ur, "I_X^{\mathbb{X}}"] \arrow[r, phantom, "\cong \Downarrow \Sigma^X"] \arrow[dr, "I_{FX}^{\mathbb{X}'}"'] & {} \\
 & \mathbb{X}'(FX,FX)
\end{tikzcd}$$

In fact, Σ^X is a single 2-cell in $\mathbb{X}'$:

$$\Sigma^X : F(1_X) \xrightarrow{\cong} 1_{F(X)} .$$

The data in 1), 2), 3) are required to satisfy the conditions 4), 5) and 6).

Note, for 4) below, that Σ^U $(U = (X,Y,Z))$ has the following type of components: with $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\Sigma_{f,g} = \Sigma^U_{f,g} : (Fg)(Ff) \xrightarrow{\cong} F(gf) ;$$

here, $Ff = F_{X,Y}(f)$; $(Fg)(Ff) = \otimes'_{FU}(Ff, Fg)$;

etc.

4) For every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbb{X}$;

the following commutes:

(*)

$$\begin{tikzcd}
((Fh)(Fg))(Ff) \arrow[rr, "\alpha'_{Ff,Fg,Fh}"] \arrow[d, "(\Sigma_{g,h})Ff"'] & & (Fh)((Fg)(Ff)) \arrow[d, "(Fh)\Sigma_{f,g}"] \\
F(hg)(Ff) \arrow[d, "\Sigma_{f,hg}"'] & = & (Fh)F(gf) \arrow[d, "\Sigma_{gf,h}"] \\
F((hg)f) \arrow[rr, "F(\alpha_{f,g,h})"'] & & F(h(gf))
\end{tikzcd}$$

Here, $\Sigma_{f,g}$ is $(\Sigma^{\mathcal{U}})_{f,g}$, with $\mathcal{U} = (X, Y, Z)$;

similarly for $\Sigma_{g,h}$, etc.

The diagram is one of objects and arrows in the category $\mathbb{X}'(FX, FW)$.

5) For $Y \xrightarrow{1_Y} Y \xrightarrow{g} Z$ in $\mathbb{X}$:

$$
\begin{tikzcd}
FY \arrow[r, bend left, "F(1_Y)"] \arrow[r, bend right, "1_{FY}"'] \arrow[r, phantom, "\Downarrow F_Y"] \arrow[rr, bend right=60, "Fg"'] & FY \arrow[r, "Fg"] \arrow[d, phantom, "\Downarrow \rho_{Fg}"] & FZ \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
FY \arrow[r, bend left, "F(1_Y)"] \arrow[rr, bend right=30, "F(g1_Y)"'] \arrow[rr, bend right=70, "Fg"'] & FY \arrow[r, "Fg"] \arrow[d, phantom, "\Downarrow F_{1_Y,g}"] & FZ \\
& \Downarrow F(\rho_g) &
\end{tikzcd}
$$

i.e.,

$$\rho_{Fg} \circ ((Fg)(F_Y)) = F(\rho_g) \circ F_{1_Y, g}$$

6) For $X \xrightarrow{f} Y \xrightarrow{1_Y} Y$ in $\mathbb{X}$; ...

'same' for λ

End of definition of 'arrow' in **Hom**

NB The definition of a morphism $F : \mathbb{X} \to \mathbb{X}'$ in <u>Hom</u> is restated, with reference to "An application ...", as follows. $F = (F; \Sigma^{\otimes}, \Sigma^{I}) = (F; (\Sigma^{U})_{U \in \mathbb{X}_0^3}, (\Sigma^{X})_{X \in \mathbb{X}_0})$

where we have: 1)* a morphism $F : \mathbb{X} \longrightarrow \mathbb{X}'$ of the underlying cat-enriched 2-graphs;

2)* appropriate data Σ^{U} $(U \in \mathbb{X}_0^3)$ such that we have morphisms in <u>pFun</u>:

$$(F_U, \Sigma^U, \hat{F}_U) : (\mathbb{X}(U), \hat{\mathbb{X}}(U), \otimes^{\mathbb{X}}_U)$$

$$\longrightarrow (\mathbb{X}'(FU), \hat{\mathbb{X}}'(FU), \otimes^{\mathbb{X}'}_{FU})$$

3)* appropriate data Σ^X $(X \in \mathbb{X}_0)$ such that we have morphisms in <u>pFun</u>:

$$(\mathrm{Id}_{\mathbb{1}}, \Sigma^X, F_{X,X}) : (\mathbb{1}, \mathbb{X}(X,X))$$

$$\longrightarrow (\mathbb{1}, \mathbb{X}'(FX, FX));$$

these data are to satisfy conditions 4), 5) and 6) (unchanged).

<u>Composition</u> in Hom: use composition of morphisms of cat-enriched 2-graphs, and (!) composition in pFun.

<u>However</u>, one has to check that for:

$$\mathbb{X} \xrightarrow{F} \mathbb{X}' \xrightarrow{F'} \mathbb{X}''$$

the composition

$$\mathbb{X} \xrightarrow{F'F} \mathbb{X}''$$

so defined <u>satisfies</u> conditions 4), 5) and 6).

I will draw the diagram for condition 4).

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. The diagram on p 15 for $F'F$ in place of F is the outside hexagon of the following (the six nodes of the hexagon are put in boxes). The diagram is a a diagram of objects and arrows in the category $\mathbb{X}''(F'FX, F'FW)$.

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$$
\begin{tikzcd}
\boxed{((F'Fh)(F'Fg))(FF'f)} \arrow[rrr, "\alpha^{\mathbb{X}''}_{F'Ff,\, F'Fg,\, F'Fh}"] \arrow[d, "F'_{Fg,Fh}\, F'Ff"'] & & & \boxed{(F'Fh)((F'Fg)(F'Ff))} \arrow[d, "(F'Fh)(F'_{Ff,Fg})"'] \\
F'(FhFg)(F'Ff) \arrow[d, "F'(F_{g,h})FF'f"'] \arrow[dr, "F'_{Ff,\, FhFg}"] & {} \arrow[r, phantom, "1"] & {} & (F'Fh)(F'(Fg\,Ff)) \arrow[dl, "F'_{Fg Ff,\, Fh}"'] \arrow[d, "(F'Fh)F'(F_{f,g})"] \\
\boxed{(F'F(hg))(F'Ff)} \arrow[d, "F'_{Ff,\, F(hg)}"'] \arrow[r, phantom, "2"] & F'((FhFg)(Ff)) \arrow[r, "F'(\alpha^{\mathbb{X}'}_{Ff,Fg,Fh})"] \arrow[dl, "F'(F_{g,h}Ff)"] & F'(Fh(FgFf)) \arrow[dr, "F'(FhF_{f,g})"'] \arrow[r, phantom, "3"] & \boxed{(F'Fh)(F'F(gf))} \arrow[d, "F'_{F(gf),\, Fh}"] \\
F'(F(hg)(Ff)) \arrow[d, "F'(F_{f,hg})"'] & {} \arrow[r, phantom, "4"] & {} & F'((Fh)(F(gf))) \arrow[d, "F'(F_{gf,h})"] \\
\boxed{F'F((hg)f)} \arrow[rrr, "F'F(\alpha^{\mathbb{X}}_{f,g,h})"'] & & & \boxed{F'F(h(gf))}
\end{tikzcd}
$$

The hexagon 1 is the 'α-condition' (4), p. 15)
for the morphism $F' : \mathbb{X}' \to \mathbb{X}''$, instantiated at the triple $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{Fh} FW$ in $\mathbb{X}'$!

The hexagon 4 is the α-condition for the morphism $F : \mathbb{X} \to \mathbb{X}'$, at (f, g, h) – with an application of the functor

$$F'_{FX,FW} : \mathbb{X}'(FX, FW) \longrightarrow \mathbb{X}''(F'FX, F'FW).$$

The quadrangle 2 comes from the naturality condition for $(\Sigma')^{FX,FY,FW}$ in:

$$\begin{tikzcd}
\mathbb{X}'(FX, FY) \times \mathbb{X}'(FY, FW) \arrow[r, "\otimes^{\mathbb{X}'}"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{(\Sigma')^{FX,FY,FW}}{\Longrightarrow}"] & \mathbb{X}'(FX, FW) \arrow[d, "F'"] \\
\mathbb{X}''(F'FX, F'FY) \times \mathbb{X}'(F'FY, F'FW) \arrow[r, "\otimes^{\mathbb{X}''}"'] & \mathbb{X}''(F'FX, F'FW)
\end{tikzcd}$$

instantiated at the arrow

$$(Ff, FhFg) \xrightarrow[(1_{Ff},\, F_{g,h})]{} (Ff, F(hg)) \quad \text{in the upper left}$$

corner. For 3 : " use $(\Sigma')^{FX, FZ, FW}$.

End of definition of the category $\underline{\text{Hom}}$.

§3 Sana Bicat

§3.1 Object of $\underline{\text{Sana}}$ $\underline{\text{Bicat}}$: $\tilde{\mathbb{X}}$; consists of:

$1)_{SB}$ A category-enriched 2-graph $\mathbb{X}$ — see above; $\mathbb{X}$.

$2)_{SB}$ For any triple $\mathcal{U} \in \mathbb{X}_0^3$, a $\underline{\text{composition}}$ sana functor (object of $\underline{\text{Sona Fun}}$: see "An application ...")

$$\tilde{\otimes}_{\mathcal{U}} : \underbrace{\mathbb{X}(\mathcal{U})}_{\text{see p } 5} \xrightarrow{\text{sana}} \hat{\mathbb{X}}(\mathcal{U}) \quad ;$$

that is, a span of categories and functors

$$\begin{tikzcd}[column sep=small] & \tilde{\otimes}_{\mathcal{U}} \arrow[dl, "\Phi_{\mathcal{U}}"'] \arrow[dr, "\Psi_{\mathcal{U}}"] & \\ \mathbb{X}(\mathcal{U}) & & \hat{\mathbb{X}}(\mathcal{U}) \end{tikzcd}$$

satisfying conditions given in "An application...".

and for each $X \in \mathbb{X}_0$, an "identity"

sana functor

$$\tilde{I}_X : \mathbb{1} \xrightarrow{\text{sana}} \mathbb{X}(X,X) : \quad \begin{tikzcd}[column sep=small] & \tilde{I}_X \arrow[dl, "\Phi_X^{(I)}"'] \arrow[dr, "\Psi_X^{(I)}"] & \\ \mathbb{1} & & \mathbb{X}(X,X) \end{tikzcd}$$

— To explain what α, λ and ρ are in the sana-context, we use some abbreviated terminology.
Suppose the data in 1) & 2) are given.
Suppose $X, Y, Z \in \mathbb{X}_0$, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ are in $\mathbb{X}(X,Y) \times \mathbb{X}(Y,Z)$. Let $s \in \tilde{\otimes}_{X,Y,Z}$ — meaning that s is an object of the category $\tilde{\otimes}_{X,Y,Z}$.
The expression $g \circ_s f$ (read: "the s-composite of f and g") is <u>defined</u> iff$_{\text{def}}$ $(f,g) = \Phi_{X,Y,Z}(s)$, and if so,
$g \circ_s f \overset{\text{def}}{=} \Psi_{X,Y,Z}(s)$. Notice that, since $\Phi_{X,Y,Z}$ is

surjective on objects, for any $(f,g)\in \mathbb{X}(X,Y)\times\mathbb{X}(Y,Z)$, there is at least one s ($\in \tilde{\otimes}_{X,Y,Z}$) for which $g\circ_s f$ is defined. Writing

$$\tilde{\otimes}_{X,Y,Z}(f,g)$$

for the fiber $\Phi^{-1}_{X,Y,Z}((f,g))$ of $\Phi_{X,Y,Z}$

($\tilde{\otimes}_{X,Y,Z}(f,g)$: the subcategory of $\tilde{\otimes}_{X,Y,Z}$ with objects and arrows that map by $\Phi_{X,Y,Z}$ to the object (f,g), resp. the identity arrow on (f,g), of the category $\mathbb{X}(U)$ $(U=(X,Y,Z))$),

$g\circ_s f$ is defined iff $s\in\tilde{\otimes}_{X,Y,Z}(f,g)$.

Also note that if $g\circ_s f$ and $g'\circ_s f'$ are both defined, then in fact $g=g'$ and $f=f'$: s determines f and g as well as $g\circ_s f$, in the expression $g\circ_s f$.

With the same tricat $\tilde{\mathbb{X}}$, we have the further data 3.1), 3.2) and 3.3):

3.1)$_{SB}$: A family

$$\tilde{\alpha}=\langle\tilde{\alpha}^{X,Y,Z,W}\rangle_{(X,Y,Z,W)\in\mathbb{X}_0^4}$$

where, for a given quadruple (X, Y, Z, W),

$\tilde{\alpha}^{X,Y,Z,W}$ – simply $\tilde{\alpha}$ in what follows –

is an operation whose domain of definition, denoted $E = E^{X,Y,Z,W}$

is the set of all quadruples (s_0, s_1, s_2, s_3) such that,

for suitable (uniquely determined) f, g, h as in

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$

each of: $g \circ_{s_0} f$, $h \circ_{s_1} g$, $h \circ_{s_2} (g \circ_{s_0} f)$

and $(h \circ_{s_1} g) \circ_{s_3} f$

is defined, ~~and~~ the value at (s_0, s_1, s_2, s_3),

$\tilde{\alpha}_{s_0, s_1, s_2, s_3}$, is an arrow in $\mathbb{X}(X, W)$ of the

following kind (an isomorphism):

$$\tilde{\alpha}_{s_0,s_1,s_2,s_3} : (h \circ_{s_1} g) \circ_{s_3} f \xrightarrow{\cong} h \circ_{s_2} (g \circ_{s_0} f).$$

Furthermore, we require that $\alpha_{s_0, s_1, s_2, s_3}$ be

natural in $\vec{s} \in (s_0, s_1, s_2, s_3)$ (it does not quite make sense to say: "natural in each of the variables s_0, s_1, s_2, s_3).

In fact, E is regarded as a full subcategory of

$$\tilde{\otimes}_{X,Y,Z} \times \tilde{\otimes}_{Y,Z,W} \times \tilde{\otimes}_{X,Z,W} \times \tilde{\otimes}_{X,Y,W}$$

and we have functors

$$E = E^{X,Y,Z,W} \overset{K}{\underset{L}{\rightrightarrows}} \mathbb{X}(X,W)$$

for which
$$K(s_0,s_1,s_2,s_3) = (h \circ_{s_1} g) \circ_{s_3} f$$
$$L(s_0,s_1,s_2,s_3) = h \circ_{s_2} (g \circ_{s_0} f)$$

and the $\tilde{\alpha}^{X,Y,Z,W}$ datum is required to be a natural transformation, a natural isomorphism:

$$\tilde{\alpha}^{X,Y,Z,W} : K \xrightarrow{\cong} L.$$

Next, let us describe the items E, K, L more diagrammatically — this will also help verifying claim (**) on p. 2, the fact that Sana Bicat is dual-regular.

The composites of $\Phi_{A,B,C} : \tilde{\otimes}_{A,B,C} \longrightarrow \mathbb{X}(A,B) \times \mathbb{X}(B,C)$ with product projections are denoted, with suppressing reference to A,B,C, by

$$\Phi_0 : \tilde{\otimes}_{A,B,C} \longrightarrow \mathbb{X}(A,B)$$
$$\Phi_1 : \tilde{\otimes}_{A,B,C} \longrightarrow \mathbb{X}(B,C)$$

The category $E = E^{X,Y,Z,W}_{X,Y,Z,W}$ is defined as the limit (in Cat) of the following diagram of categories and functors:

$$
\begin{tikzcd}
 & & & \mathbb{X}(Y,Z) & & & \\
 & \tilde{\otimes}_{X,Y,Z} \arrow[urr, "\Phi_1"] \arrow[dl, "\Phi_0"'] \arrow[dr, "\Psi"] & & & & \tilde{\otimes}_{Y,Z,W} \arrow[ull, "\Phi_0"'] \arrow[dl, "\Psi"'] \arrow[dr, "\Phi_1"] & \\
\mathbb{X}(X,Y) & & \mathbb{X}(X,Z) & & \mathbb{X}(Y,W) & & \mathbb{X}(Z,W) \\
 & \tilde{\otimes}_{X,Y,W} \arrow[ul, "\Phi_0"] \arrow[urrr, "\Phi_1"] & & & & \tilde{\otimes}_{X,Z,W} \arrow[ulll, "\Phi_0"] \arrow[ur, "\Phi_1"'] &
\end{tikzcd}
$$

The limit projections

$$
\begin{tikzcd}
 & & E \arrow[dll, "\pi_{012}"'] \arrow[dl, "\pi_{123}"] \arrow[d, "\pi_{023}"] \arrow[dr, "\pi_{013}"] & \\
\tilde{\otimes}_{X,Y,Z} & \tilde{\otimes}_{Y,Z,W} & \tilde{\otimes}_{X,Z,W} & \tilde{\otimes}_{X,Y,W}
\end{tikzcd}
$$

determine the others as certain composites.

The functors

$$
\begin{tikzcd}
E \arrow[r, shift left, "K"] \arrow[r, shift right, "L"'] & \mathbb{X}(X,W)
\end{tikzcd}
$$

are

the composites in:

$$\begin{tikzcd}
 & \tilde{\otimes}_{X,Y,W} \arrow[dr, "\Psi_{X,Y,W}"] \arrow[dd, phantom, "\cong\ \Downarrow \tilde{\alpha}^{X,Y,Z,W}"] & \\
E^{X,Y,Z,W} \arrow[ur, "\pi_{013}"] \arrow[dr, "\pi_{023}"'] & & \mathcal{X}(X,W) \\
 & \tilde{\otimes}_{X,Z,W} \arrow[ur, "\Psi_{X,Z,W}"'] &
\end{tikzcd}$$

;

as indicated, $\tilde{\alpha}^{X,Y,Z,W}$ is, by definition, a natural isomorphism between the composite functors.

NB Compare the above to this:

$$\begin{tikzcd}
 & & g & & & \\
 & s_0 \arrow[ur, mapsto, "\Phi_1"] \arrow[dl, mapsto, "\Phi_0"'] \arrow[dr, mapsto, "\Psi"] & & & s_1 \arrow[ull, mapsto, "\Phi_0"'] \arrow[dl, mapsto, "\Psi"'] \arrow[dr, mapsto, "\Phi_1"] & \\
f & & g \circ_{s_0} f & h \circ_{s_1} g & & h \\
 & s_3 \arrow[ul, mapsto, "\Phi_0"] \arrow[urr, mapsto, "\Phi_1"'] \arrow[dr, mapsto, "\Psi"'] & & & s_2 \arrow[ull, mapsto, "\Phi_0"] \arrow[ur, mapsto, "\Phi_1"'] \arrow[dl, mapsto, "\Psi"] & \\
 & & (h \circ_{s_1} g) \circ_{s_3} f \arrow[r, "\tilde{\alpha}_{s_0,s_1,s_2,s_3}"'] & h \circ_{s_2} (g \circ_{s_0} f) & &
\end{tikzcd}$$

$(3.2)_{\underline{SB}}$ A family

$$\lambda = \langle \lambda^{X,Y} \rangle_{(X,Y)\in \mathbb{X}_0^{\,2}}$$

such that $\lambda = \lambda^{X,Y}$ is a mapping with domain of definition the set of pairs (u, s) such that $u \in \tilde{\hat{I}}_X$, and, with $(1_Y)_u \underset{\mathrm{def}}{=} \Psi^{(I)}_Y(u)$, we have $s \in \tilde{\hat{\otimes}}_{X,Y,Y}(f, (1_Y)_u)$, and its value, denoted $\lambda_{u,s}$ is an arrow: $\lambda_{u,s} : (1_Y)_u \circ_s f \xrightarrow{\cong} f$

in $\mathbb{X}(X,Y)$:

$$\begin{tikzcd} & Y \arrow[dr, "(1_Y)_u"] & \\ X \arrow[ur, "f"] \arrow[rr, "(1_Y)_u \circ_s f"] \arrow[rr, bend right=50, "f"'] & \arrow[phantom, d, "\Downarrow \lambda_{u,s}"] & Y \\ & \, & \end{tikzcd}$$

It is required that $\lambda_{u,s}$ be natural in (u, s).

$(3.3)_{\underline{SB}}$ (abbreviated)

$$\rho = \langle \rho^{Y,Z} \rangle_{(Y,Z)\in \mathbb{X}_0^{\,2}}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ Y \arrow[ur, "(1_Y)_u"] \arrow[rr, "g \circ_t (1_Y)_u"] \arrow[rr, bend right=50, "g"'] & \arrow[phantom, d, "\Downarrow \rho_{u,t}"] & Z \\ & \, & \end{tikzcd}$$

$\rho_{u,t}$: natural in (u, t).

I need some more notation to state the conditions the above data are to satisfy.

Assuming we have the above, let $A, B, C \in \mathbb{X}_0$ and consider the span

$$\begin{tikzcd}
& \tilde{\otimes}_{A,B,C} \arrow[dl, "\bar{\Phi}_{A,B,C}"'] \arrow[dr, "\Psi_{A,B,C}"] & \\
\mathbb{X}(A,B) \times \mathbb{X}(B,C) & & \mathbb{X}(A,C)
\end{tikzcd}$$

Let $\mu : f \to f'$ be an arrow in $\mathbb{X}(A,B)$, $\nu : g \to g'$ one in $\mathbb{X}(B,C)$. Let $s \in \tilde{\otimes}_{A,B,C}(f,g)$, $s' \in \tilde{\otimes}_{A,B,C}(f',g')$ so that $g \circ_s f$ and $g' \circ_{s'} f'$ are defined ($g \circ_s f = \Psi_{A,B,C}(s)$, $g' \circ_{s'} f' = \Psi_{A,B,C}(s')$). Since $\bar{\Phi}_{A,B,C}$ is full and faithful, there is a unique

$\rho^{s,s'}_{\mu,\nu} =$ $\rho : s \to s'$ such that $\bar{\Phi}_{A,B,C}(\rho) = (\mu, \nu)$, the arrow $(\mu, \nu) : (f, g) \to (f', g')$ in $\mathbb{X}(A,B) \times \mathbb{X}(B,C)$.

We denote the arrow in $\mathbb{X}(A,C)$:

$$\Psi_{A,B,C}(\rho) : g \circ_s f \longrightarrow g' \circ_{s'} f'$$

by :

$$\nu \circ_{s,s'} \mu : g \circ_s f \longrightarrow g' \circ_{s'} f'.$$

In case $\mu = 1_f : f \to f'$, I write $\nu \circ_{s,s'} f$ for $\nu \circ_{s,s'} \mu$; and similarly when ν is an identity.

— · —

Before getting to the (familiar-looking) "Mac Lane conditions", let me express the naturality of $\tilde{\alpha}$ (see: p. 27) in the just-introduced notation:

Given

$$\begin{tikzcd} X \arrow[r, shift left=1.5, "f"] \arrow[r, shift right=1.5, "\hat{f}"'] \arrow[r, phantom, "\Downarrow\varphi"] & Y \arrow[r, shift left=1.5, "g"] \arrow[r, shift right=1.5, "\hat{g}"'] \arrow[r, phantom, "\Downarrow\gamma"] & Z \arrow[r, shift left=1.5, "h"] \arrow[r, shift right=1.5, "\hat{h}"'] \arrow[r, phantom, "\Downarrow\eta"] & W \end{tikzcd}$$

and s_0, s_1, s_2, s_3; $\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3$ $\Big\}$ (*)

such that all the objects in the following diagram are defined, we have that the diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=large] (h \circ_{s_1} g) \circ_{s_3} f \arrow[r, "\tilde{\alpha}_{s_0,s_1,s_2,s_3}"] \arrow[d, "(\eta \circ_{s_1,\hat{s}_1} \gamma) \circ_{s_3,\hat{s}_3} \varphi"'] \arrow[dr, phantom, "="] & h \circ_{s_2} (g \circ_{s_0} f) \arrow[d, "\eta \circ_{s_2,\hat{s}_2} (\gamma \circ_{s_0,\hat{s}_0} \varphi)"] \\ (\hat{h} \circ_{\hat{s}_1} \hat{g}) \circ_{\hat{s}_3} \hat{f} \arrow[r, "\tilde{\alpha}_{\hat{s}_0,\hat{s}_1,\hat{s}_2,\hat{s}_3}"'] & \hat{h} \circ_{\hat{s}_2} (\hat{g} \circ_{\hat{s}_0} \hat{f}) \end{tikzcd}$$

The above data 1), 2), 3) are required to satisfy the conditions 4.1), 4.2), 4.3):

4.1)$_{SB}$ Given

$$
\begin{tikzcd}
 & & Y \arrow[r, "g"] & Z \arrow[r, "h"] & W \arrow[dr, "i"] & \\
 & X \arrow[ur, "f"] & & & & V
\end{tikzcd}
$$

in $\mathbb{X}$, and given s_i for $i \in \{0, 1, \dots, 11\}$ so that all expressions $u \circ_{s_i} v$ in the following diagram are defined, the diagram commutes. As a further abbreviation, I write $u \circ_i v$ for $u \circ_{s_i} v$, and similarly for $v \circ_{i,j} f$, $\alpha_{i,j,h,k}$ ...

$$
\begin{tikzcd}
 & & (i \circ_6 h) \circ_7 (g \circ_0 f) \arrow[drr, "\tilde{\alpha}_{2,6,4,7}"] & & \\
((i \circ_6 h) \circ_8 g) \circ_9 f \arrow[urr, "\tilde{\alpha}_{0,8,7,9}"] \arrow[dr, "\tilde{\alpha}_{1,6,10,8} \circ_{9,11} f"'] & & = & & i \circ_4 (h \circ_2 (g \circ_0 f)) \\
 & (i \circ_{10} (h \circ_1 g)) \circ_{11} f \arrow[rr, "\tilde{\alpha}_{3,10,5,4}"'] & & i \circ_5 ((h \circ_1 g) \circ_3 f) \arrow[ur, "i \circ_{5,4} \tilde{\alpha}_{0,1,2,3}"'] &
\end{tikzcd}
$$

4.2) $\underline{SB}$ (abbreviated):

$$\begin{tikzcd} & Y \arrow[r, "(1_Y)_u"] & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & & Z \end{tikzcd}$$

$$\begin{tikzcd} (g \circ_{s_1} (1_Y)_u) \circ_{s_3} f \arrow[rr, "\alpha_{s_0,s_1,s_2,s_3}"] \arrow[dr, "\rho_{u,s_1} \circ_{s_3,s_4} f"'] \arrow[rr, phantom, shift right=6ex, "="] & & g \circ_{s_2} ((1_Y)_u \circ_{s_0} f) \arrow[dl, "g \circ_{s_2,s_4} \lambda_{u,s_0}"] \\ & g \circ_{s_4} f & \end{tikzcd}$$

4.3) $\underline{SB}$ (abbreviated):

$$\begin{tikzcd} & X \arrow[dr, "(1_X)_v"] & \\ X \arrow[ur, "(1_X)_v"] \arrow[rr, "(1_X)_v \circ_s (1_X)_v" description] \arrow[rr, bend right=50, "(1_X)_v"'] \arrow[rr, phantom, shift right=4ex, "\lambda_{v,s}\Downarrow \; = \; \Downarrow\rho_{v,s}"] & & X \end{tikzcd}$$

$$\boxed{\lambda_{v,s} = \rho_{v,s}}$$

end of definition of a sana-bicat: <u>object</u> of <u>SanaBicat</u>

§ 3.2 Morphism in $\underline{\text{Sana Bicat}}$

Morphism of sana bicategories:

$$(F, \sigma^{\otimes}, \sigma^{I}) : (\mathbb{X}, \tilde{\otimes}, \tilde{I}, \alpha, \lambda, \rho)$$

$$\longrightarrow (\mathbb{X}', \tilde{\otimes}', \tilde{I}', \alpha', \lambda', \rho')$$

consists of:

1) $F : \mathbb{X} \longrightarrow \mathbb{X}'$: morphism of category-enriched 2-graphs (as in $\underline{\text{Hom}}$);

2) $\sigma^{\otimes} = \langle \sigma_U \rangle_{U \in \mathbb{X}_0^3}$, where each σ_U is a a functor $\sigma_U : \tilde{\otimes}_U \longrightarrow \tilde{\otimes}'_{FU}$ such that σ_U, together with F_U and $\hat{F}_U$, make up a morphism

$$(F_U, \sigma_U, \hat{F}_U) : \tilde{\otimes}_U \longrightarrow \tilde{\otimes}'_{FU}$$

of sana functors;

$$\begin{tikzcd}
 & \tilde{\otimes}_U \arrow[dl, "\Phi_U"'] \arrow[dr, "\Psi_U"] \arrow[rrr, "\sigma_U"] & & & \tilde{\otimes}'_{FU} \arrow[dl, "\Phi'_{FU}"'] \arrow[dr, "\Psi'_{FU}"] & \\
\mathbb{X}(U) \arrow[rrr, dashed, "F_U"] & & \hat{\mathbb{X}}(U) \arrow[rrr, "\hat{F}_U"'] & \mathbb{X}'(FU) & & \hat{\mathbb{X}}'(FU)
\end{tikzcd}$$

i.e., a (strict) morphism of spans:

$$F_U \circ \Phi_U = \Phi'_{FU} \circ \sigma_U ,$$
$$\hat{F}_U \circ \Psi_U = \Psi'_{FU} \circ \sigma_U$$

(see: "An application ...").

3) $\sigma^I = \langle \sigma_X \rangle_{X \in \mathbb{X}_0}$:

$$(!, \sigma_X, F_{X,X}) : \tilde{I}_X \longrightarrow \tilde{I}_{FX}$$

a morphism in Sana Fun.

The above data are required to preserve α, λ, ρ:

4) For $\tilde{\alpha}$: for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$

and $\tilde{\alpha}_{s_0, s_1, s_2, s_3} : (h \circ_{s_1} g) \circ_{s_3} f \longrightarrow h \circ_{s_2} (g \circ_{s_0} f)$

in the category $\mathbb{X}(X, W)$, I abbreviate

$$s_0' = \sigma_{X,Y,Z}(s_0)$$

$$s_1' = \sigma_{Y,Z,W}(s_1)$$

$$s_2' = \sigma_{X,Z,W}(s_2)$$

$$s_3' = \sigma_{X,Y,W}(s_3) \;;$$

by what we already know, we have that

$$F((h \circ_{s_1} g) \circ_{s_3} f) = (Fh \circ_{s_1'} Fg) \circ_{s_3'} Ff$$

(where, of course, Ff abbreviates $F_{X,Y}(f)$, etc.),

and similarly for the other triple composite;

the new requirement is

$$(*) \qquad \boxed{F_{X,W}(\alpha_{s_0,s_1,s_2,s_3}) = \alpha'_{s_0',s_1',s_2',s_3'}}$$

5), 6) Entirely similar preservation requirements are in place for λ and ρ.

> end of definition of morphism in Sara bicat's and of the category SaraBicat