Category $\mathbb{A}$ is $\boxed{\text{dual-regular}}$ if

$$\mathbb{A} \simeq \mathrm{Reg}(R, Set) \qquad (1)$$

for some small regular category $R$.

NB For $\mathbb{A}$ dual-regular,

$$\mathbb{A}^{\#} \stackrel{\text{def}}{=} \text{Filt Colim -small Prod}(\mathbb{A}, Set)$$

can be chosen for $R$ in (1). In fact, if (1) holds, then $\mathbb{A}^{\#}$ is the (Barr-) exact completion of $R$ (MM, "....Barr exact...", APAL 47 (199

Alternatively: $\mathbb{A}$ is dual-regular if it is the category of models of a regular theory $(T, \wedge, \exists)$ or, equivalently, of a finite-limit-regular-epi sketch.

Examples: the doctrines (Kock/Reyes 1977) of (finitary) categorical logic: categories of structured categories defined by (finitary) universal properties and exactness conditions.

E.g.:

**Prod** : cat's with finite products (Lawvere algebraic theories)

**Lex** : categories having finite limits

**Reg** : regular categories

**Pretop** : pretoposes

**Top** : elementary toposes

Showing these dual-regular is straightforward — but also see my "Generalized sketches..." (MM, three papers in JPAA, 1997)

NB: these categories are not essentially algebraic (locally finitely presentable).

With Ross Street (Cahiers TCDC, 1980)

I denote by $\boxed{\underline{\text{Hom}}}$ the category of (small) bicategories and homomorphisms (= strong morphisms) ("small": relative to an arbitrary choice of a Grothendieck universe).

$\boxed{\text{Theorem}}$ $\underline{\text{Hom}}$ is a dual-regular category.

$\underline{\text{Hom}}^{\#}$ (dual; see p. 1) is equivalent to a countable, recursively presented exact category.
$\underline{\text{Hom}}$ is $\aleph_1$-accessible, and it has ($\aleph_0$-) filtered colimits.

$\boxed{\text{Corollary}}$ (of the proof).
The category $\underline{\text{Mon}}$ of (small) monoidal categories and strong monoidal functors is dual-regular.

<u>pFun</u>, the category of free-living functors, with pseudo morphisms:

object: $(X, A, \Phi)$: $X, A$ small categories

$\Phi : X \to A$ (functor)

morphism: $(X, A, \Phi) \xrightarrow{(\tau, \theta, \Sigma)} (X', A', \Phi')$

$$
: \begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\cong\ \Downarrow\Sigma"] & A \arrow[d, "\theta"] \\ X' \arrow[r, "\Phi'"'] & A' \end{tikzcd}
$$

$$\Sigma : \Phi'\tau \xrightarrow{\cong} \theta\Phi$$

composition: pasting.

'Key' to all:

$\boxed{\text{Proposition 1}}$ p<u>Fun</u> is dual-regular.

Let: $T$ : small 2-category whose underlying category is freely generated by a graph.

$\boxed{Ps(T, Cat)}$ : the category:

object : 2-functor $T \xrightarrow{M} Cat$

morphism : pseudo-natural transformation

$h : M \to N$ $\qquad$ $T \underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow h}}}} Cat$ (= 2-cat of small cat's)

**Proposition 2** (John Bourke)

$Ps(T, Cat)$, with $T$ above, is accessible with filtered colimits.

**Proposition $2^{bis}$**

$Ps(T, Cat)$, with $T$ above is dual-regular.

John's proof uses Proposition 1 above, and employs the Limit theorem of MM - Bob Paré "Accessible categories..." (1989, p 106).

For the proof of Prop $2^{bis}$, see later.

Prop 1 is the special case when

$$T :=: \quad \boxed{0 \xrightarrow{2} 1}$$

**Examples:**

$$T_{ps} :\overset{def}{=}:$$

$$\begin{tikzcd}
 & 1 \arrow[dr, "6"] \arrow[d, phantom, "\cong\ \Downarrow 8"] & \\
0 \arrow[ur, "4"] \arrow[dr, "5"'] & {} & 3 \\
 & 2 \arrow[ur, "7"'] &
\end{tikzcd}$$

$$8 : 6 \circ 4 \xrightarrow{\cong} 7 \circ 6$$

PS = the category of "pseudo-squares" $\overset{def}{=}$ $Ps(T_{ps})$

$$T_{cs} :\overset{def}{=}:$$

$$\begin{tikzcd}
 & 1 \arrow[dr, "6"] \arrow[d, phantom, "="] & \\
0 \arrow[ur, "4"] \arrow[dr, "5"'] & {} & 3 \\
 & 2 \arrow[ur, "7"'] &
\end{tikzcd}$$

$$6 \circ 4 = 7 \circ 6$$

CS = the category of "commutative squares" $\overset{def}{=}$ $Ps(T_{cs})$

$T_{ps}$ does, $T_{cs}$ does not satisfy John Bourke's freeness condition (see Prop 2)

Facts: $\boxed{(i)}$ CS is a full subcategory of PS.

$\boxed{(ii)}$ The inclusion $CS \hookrightarrow PS$ is not an equivalence (there are $X \in PS$ not isomorphic in PS to any commutative square).

$\boxed{(iii)}$ Every pseudo-square is a retract of some commutative square: for every $X \in PS$, there are, in PS:

$$\begin{tikzcd} \hat{X} \arrow[d, shift right, "p"'] \\ X \arrow[u, shift right, "i"'] \end{tikzcd} \qquad \hat{X} \in CS, \quad pi = 1_X.$$

I.e.: the idempotent-splitting completion of CS is (equivalent to) PS.

Hence: $\boxed{(iv)}$ CS is not accessible, but its i.p.s. completio (Karoubi envelope) is accessible (and more).

$\boxed{(v)}$ Since $Ps(T, Cat)$, for any 2-cat $T$, is naturally a 2-category (2-cells: modifications), PS and CS are 2-cats.

We have: $CS \hookrightarrow PS$ is a biequivalence.

There are generalizations of the Facts in which $T$ and $T'$ are replaced by appropriate

finite-pseudo-limit sketches.

On the <u>proofs</u> :

<u>Definition</u> (MM, "Avoiding the axiom of choice...", JPAA 108 (1996))

For categories $X$ and $A$, a <u>saturated</u> <u>ana</u><u>functor</u> (<u>sana</u>-<u>functor</u> for short) from $X$ to $A$

$$F : X \xrightarrow{\text{sana}} A$$

is a span:

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

of categories and functors <u>such</u> <u>that</u>

1) $\varphi$ is a "trivial fibration":
   - 1.1) $\varphi$ is surjective on objects,
   - 1.2) $\varphi$ is full and faithful;

2) $(\varphi^*, \psi^*) : |F|^* \longrightarrow X^* \times A^*$ is a discrete opfibration

($\mathbb{C}^*$ : the underlying groupoid of the cat $\mathbb{C}$).

Category <u>Sana</u> <u>Fun</u> :

objects: sanafunctors (w/ variable domain/codomain)

morphism: strict morphism of spans:

for $F = (|F|, X, A, \varphi, \psi)$, $F' = (|F'|, \dots)$

$(\theta, \sigma, \tau) : F \longrightarrow F'$ ::

$$\begin{tikzcd}
& {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[ddd, "\sigma"] & \\
X \arrow[d, "\tau"'] \arrow[r, phantom, "="] & {} & A \arrow[d, "\theta"] \arrow[l, phantom, "="] \\
X' & & A' \\
& {|F'|} \arrow[ul, "\varphi'"] \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

:: (strictly) commutative.

Inspection (via a regular theory): <u>Sana</u> <u>Fun</u> is dual-regular. In fact all conditions but 1.1 (surjectivity on objects) are essentially algebraic.

Proposition $1^{bis}$ $\quad \underline{\mathrm{p\,Fun}} \simeq \underline{\mathrm{Sana\,Fun}}$.

On the proof:

Definition A split sana-functor is an entity

$$(F, \Phi, \mu)$$

where: $F = (|F|, X, A, \varphi, \psi) \in \underline{\mathrm{Sana\,Fun}}$

$$X \xrightarrow{\Phi} A \quad (\in \mathrm{p}\,\underline{\mathrm{Fun}})$$

and:

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\mu"] & \\ X \arrow[rr, "\Phi"'] & {} & A \end{tikzcd} \qquad \mu : \Phi\varphi \xrightarrow{\cong} \psi$$

(isomorphism)

A <u>morphism</u> of split sana functors :

$$(\tau, \theta, \sigma, \Sigma) : (F, \Phi, \mu) \longrightarrow (F', \Phi', \mu')$$

such that :

1) $(\tau, \theta, \sigma) : F \longrightarrow F' \in \mathrm{Arr}(\underline{\text{Sana Fun}})$

2) $(\tau, \theta, \Sigma) : \Phi \longrightarrow \Phi' \in \mathrm{Arr}(\mathrm{p}\underline{\mathrm{Fun}})$

and : 3)

$$
\begin{tikzcd}
 & F \arrow[dl] \arrow[dr] \arrow[d, phantom, "\mu"] & \\
X \arrow[rr, "\Phi"'] \arrow[d, "\tau"'] & {} & A \arrow[d, "\theta"] \\
X' \arrow[rr, "\Phi'"'] \arrow[rr, phantom, bend left=40, "\Sigma"] & & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & F \arrow[d, "\sigma"] & \\
 & F' \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] \arrow[d, phantom, "\mu'"] & \\
X' \arrow[rr, "\Phi'"'] & {} & A'
\end{tikzcd}
$$

$$\boxed{\mu'\sigma = \mu(\Sigma^{-1}\varphi)}$$

We have the category: $\underline{\text{split}}$ $\underline{\text{Sana Fun}}$

Proposition $1^{\text{bis bis}}$ The forgetful functors $R$ and $S$ in:

$$\begin{tikzcd} & \underline{\text{split}}\ \underline{\text{SanaFun}} \arrow[dl, "S"'] \arrow[dr, "R"] & \\ \underline{\text{SanaFun}} & & \text{p}\underline{\text{Fun}} \end{tikzcd}$$

are both full and faithful, and surjective on objects — hence, they are equivalence functors.
(thus we obtain Proposition $1^{\text{bis}}$ and Proposition 1).

(The special case when restricting to arrows $(\tau, \theta, \sigma, \Sigma)$ with $\tau, \theta$ identity functors, thus talking about an isomorphism natural transformation

$$\begin{tikzcd} X \arrow[r, bend left, "\Phi"] \arrow[r, bend right, "\Phi'"'] \arrow[r, phantom, "\downarrow\Sigma"] & A \end{tikzcd},$$

appears in the paper "Avoiding ...".)

## On R being surjective on objects:

task: given $(X, A, \Phi) \in p\,\mathrm{Fun}$

find $(F, \Phi, \mu) \in$ split SanaFun

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, "\Phi"'] \arrow[rr, phantom, shift left=4, "\mu"] & & A \end{tikzcd}$$

such that

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

is a sana-functor

Construction: $(F, \varphi, \psi)$ the 'pseudo-', or 'up-to-isomorphism' graph of the functor $\Phi$.

The construction categorifies to 2, and even 3 dimensions — usefully (2008 Oct: Montreal; 2009: June; Cape Town).

object of $|F|$: $(x, a, \nu)$ : $x \in X$ (object)

$a \in A$, $\nu : \Phi(x) \xrightarrow[\text{iso}]{\cong} a$

$$(x, a, \nu) \overset{\varphi}{\longmapsto} x$$
$$(x, a, \nu) \overset{\psi}{\longmapsto} a$$
(forgetful)

arrow in $|F|$:

$$
\begin{tikzcd}
(x, a, \nu) \arrow[d, "{(f,h)}"'] \\
(y, b, \rho)
\end{tikzcd}
\quad :: \quad
\begin{tikzcd}
x \arrow[d, "f"'] & a \arrow[d, "h"'] \\
\underbrace{y}_{\text{in } X} & \underbrace{b}_{\text{in } A}
\end{tikzcd}
\quad \text{s.t.} \quad
\begin{tikzcd}
\Phi(x) \arrow[r, "\nu"] \arrow[d, "\Phi(f)"] \arrow[dr, phantom, "\circeq"] & a \arrow[d, "h"] \\
\Phi(y) \arrow[r, "\rho"'] & b
\end{tikzcd}
$$

composition: component-wise.

$$(f, h) \overset{\varphi}{\longmapsto} f$$

$$(f, h) \overset{\psi}{\longmapsto} h$$

Main observation: $F = (|F|, X, A, \varphi, \psi)$ is a sana-functor.

---

For $\mu$: $\quad \mu : \Phi\varphi \xrightarrow{\cong} \psi$ has components

$$\mu_{(x,a,\nu)} = \nu.$$

S being surjective on objects:

a cleavage (splitting) construction

$$\begin{tikzcd}[column sep=small] & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd} \longmapsto \begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\mu"] & \\ X \arrow[rr, dashed, "\Phi"'] & {} & A \end{tikzcd}$$

---

The facts that R & S are full and faithful can be summarized by saying that, in

$$(\tau, \theta, \sigma, \Sigma) : (F, \Phi, \mu) \longrightarrow (F', \Phi', \mu')$$

an arrow of <u>splitSanaFun</u>, the components $\sigma$ and $\Sigma$ determine each other uniquely when all other components are kept fixed.

This is the main point in the proof of Proposition $\underline{1}^{\text{bis bis}}$.

On the proof the Theorem on $\underline{\mathrm{Hom}}$:

We apply Proposition $1^{\text{bisbis}}$ as a lemma,

and elaborate. Define: span of categories and functors:

$$
\begin{tikzcd}
 & \underline{\text{split Sana Bicat}} \arrow[dl, "S"'] \arrow[dr, "R"] & \\
\underline{\text{Sana Bicat}} & & \underline{\mathrm{Hom}}
\end{tikzcd}
$$

$\underline{\mathrm{Hom}}$: an object, of course, consists of

1) a category-enriched 2-graph $\mathbb{X}$:

$\mathbb{X}_0$: a set (of 0-cells)

for $X, Y \in \mathbb{X}_0$: a category $\mathbb{X}(X,Y)$;

2) for $X, Y, Z \in \mathbb{X}_0$: composition and identity

functors $\otimes_{X,Y,Z} : \mathbb{X}(X,Y) \times \mathbb{X}(Y,Z) \longrightarrow \mathbb{X}(X,Z)$,

$I_X : \mathbb{1} \longrightarrow \mathbb{X}(X,X)$;

3) $\alpha, \lambda, \rho$: the usual coherence natural isomorphisms;

plus

4) usual conditions (Mac Lane pentagon, ...).

In an object $T$ of <u>SanaBicat</u>, a <u>sana</u> <u>bicat</u>egory, we again have a cat-enriched 2-graph $\mathbb{X}$ ("the same as before"). For composition and identities, we have sana functors:

$$T^{\otimes}_{X,Y,Z} : \mathbb{X}(X,Y) \times \mathbb{X}(Y,Z) \xrightarrow{\text{sana}} \mathbb{X}(X,Z)$$

$$\begin{tikzcd}[column sep=small]
 & {|T^{\otimes}_{X,Y,Z}|} \arrow[dl, "\varphi^{\otimes}_{X,Y,Z}"'] \arrow[dr, "\psi^{\otimes}_{X,Y,Z}"] & \\
\mathbb{X}(X,Y) \times \mathbb{X}(Y,Z) & & \mathbb{X}(X,Z)
\end{tikzcd}$$

---

$$T^{I}_{X} : \mathbb{1} \xrightarrow{\text{sana}} \mathbb{X}(X,X).$$

The ana-associativity isomorphism $\alpha$ in $T$ will be like this:

Start with objects and arrows in $\mathbb{X}$ (in T):

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

Let $s_0 \in T^{\otimes}_{X,Y,Z}$ such that $\varphi^{\otimes}_{X,Y,Z}(s_0) = (f,g)$

(write: $s_0 \in T^{\otimes}_{X,Y,Z}(f,g)$ )

(such $s_0$ exists, since $\varphi^{\otimes}_{X,Y,Z}$ is surjective)

and write

$$g \circ_{s_0} f \underset{\text{def}}{=} \varphi^{\otimes}_{X,Y,Z}(s_0) : X \longrightarrow Z$$

Analogously:
let: $s_1 \in T_{Y,Z,W}(g,h)$, $h \circ_{s_1} g : Y \longrightarrow W$

$s_2 \in T_{X,Z,W}(g \circ_{s_0} f, h)$,

$$h \circ_{s_2} (g \circ_{s_0} f) : X \longrightarrow W$$

$s_3 \in T_{X,Y,W}(f, h \circ_{s_1} g)$,

$$(h \circ_{s_1} g) \circ_{s_3} f : X \longrightarrow W$$

By definition, the sana bicategory $\mathbb{T}$ will be endowed with an operation

$$(s_0, s_1, s_2, s_3) \longmapsto \alpha_{s_0,s_1,s_2,s_3} \quad , \quad \text{the latter}$$

an isomorhism in the category $\mathbb{X}(X, W)$:

$$\begin{tikzcd}
X \arrow[rr, bend left, "(h\circ_{s_1} g)\circ_{s_3} f"] \arrow[rr, bend right, "h\circ_{s_2}(g\circ_{s_0} f)"'] \arrow[rr, phantom, "\cong \Downarrow \alpha_{s_0,s_1,s_2,s_3}"] & & W
\end{tikzcd}$$

The notion of "(s)ana-bicategory" is spelled out in my ana-functor paper "Avoiding the axiom of choice..."; also, the notion of morphism of such, which is the straight structure-preserving variety. The latter fact is crucial for having <u>Sana Bicat</u> dual-regular.

An object $\mathcal{U}$ of $\underline{\text{split Sana Bicat}}$ has a part, $\mathbb{X} = R(\mathcal{U})$, a bicategory, and another, $T = S(\mathcal{U})$, a sana bicategory. In the definition of what $\mathcal{U}$ should be like, one writes down, among others, the condition making the $\alpha$ of $T$, $\alpha^{(T)}$ and the $\alpha$ of $\mathbb{X}$, $\alpha^{(\mathbb{X})}$, compatible.

We will have the *real* theorem on $\underline{\text{Hom}}$:

**Theorem** bis bis The span

$$\begin{tikzcd}
 & \underline{\text{split Sana Bicat}} \arrow[dl, "S"'] \arrow[dr, "R"] & \\
\underline{\text{SanaBicat}} & & \underline{\text{Hom}}
\end{tikzcd}$$

is a span of trivial fibrations: $R$ and $S$ are both full and faithfull, and surjective on objects

Return to Proposition $2^{bis}$, the special case for

$$T_{ps} :=: \begin{tikzcd} & 1 \arrow[dr] \arrow[dd, phantom, "\Downarrow\cong"] & \\ 0 \arrow[ur] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd},$$

$PS \underset{def}{=} Ps(T_{ps}, Cat)$ = the category of pseudo-commutative squares

To prove that PS is dual-regular, not just accessible, it seems necessary to talk about composing sana functors, to go along with the compositions of functors inherent in objects of PS.

Let $X \underset{sana}{\xrightarrow{F}} M \underset{sana}{\xrightarrow{G}} A$.

In "Avoiding ...", a composite

$$X \underset{sana}{\xrightarrow{G \circ F}} A$$

is defined and investigated.

We have:

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi^F"'] \arrow[dr, "\psi^F"] & & {|G|} \arrow[dl, "\varphi^G"'] \arrow[dr, "\psi^G"] & \\ X & & M & & A \end{tikzcd}$$

Take the usual composite of spans:

$$
\begin{tikzcd}
 & & P \arrow[dl, "\hat{\varphi}"'] \arrow[dr, "\hat{\psi}"] \arrow[d, phantom, "\mathrm{pb}"] & & \\
 & {|F|} \arrow[dl, "\varphi^F"'] \arrow[dr, "\psi^F"] & {} & {|G|} \arrow[dl, "\varphi^G"'] \arrow[dr, "\psi^G"] & \\
X & & M & & A
\end{tikzcd}
$$

There is no problem of the left leg $\varphi^F \circ \hat{\varphi}$ being a trivial fibration. However, the other condition (discrete iso opfibration) does not hold; we need to take a quotient $P/\sim$ of $P$ under an equivalence relation $\sim$, defined on objects as follows.

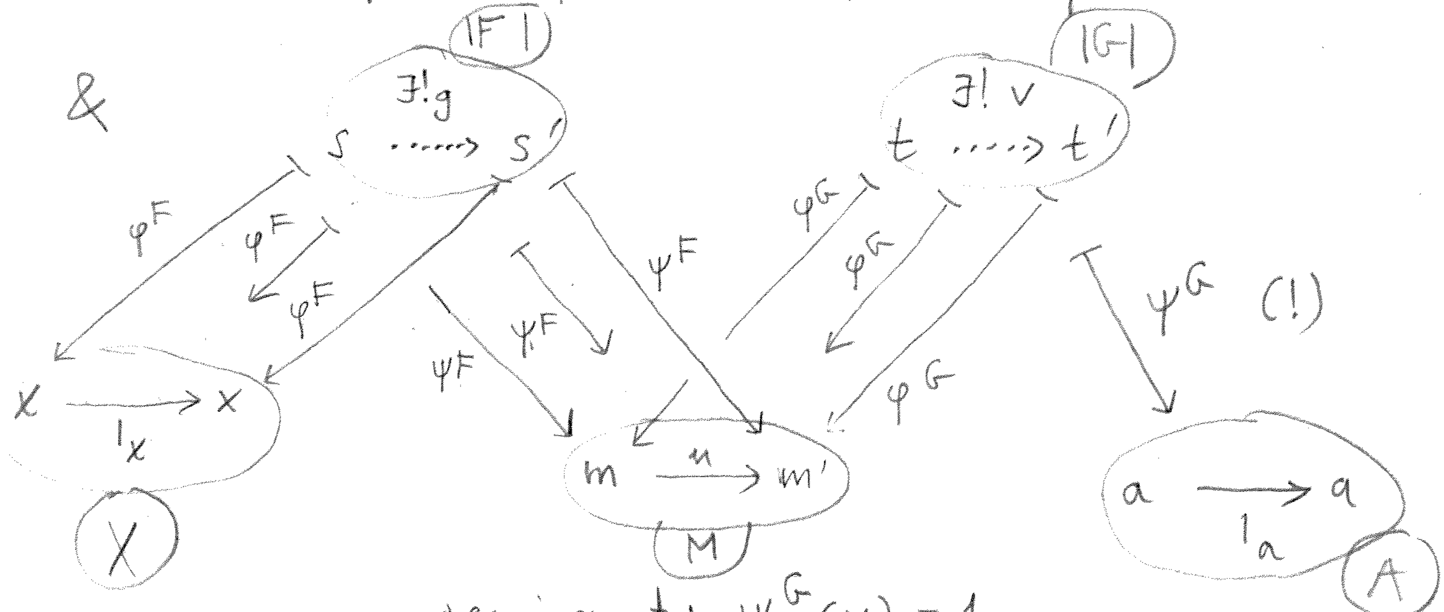
The objects of $P$ are pairs $(s,t) \in |F| \times |G|$ such that $\psi^F(s) = \varphi^G(t)$. Let $(s,t), (s',t') \in P$.

Then

$$(s,t) \sim (s',t') \quad \overset{\text{def}}{\Longleftrightarrow}$$

$$(x=)\,\varphi^F(s) = \varphi^F(s') \quad \& \quad \psi^G(t) = \psi^G(t')\,(=a)$$

&



requirement: $\psi^G(v) = 1_a$

It is easy to see that $\sim$ is an equivalence relation; a definable equivalence relation!

For the composite $G \circ F : X \xrightarrow[\text{sana}]{} A$,

$$\begin{tikzcd} & {|G\circ F|} \arrow[dl, "\tilde{\varphi}"'] \arrow[dr, "\tilde{\psi}"] & \\ X & & A \end{tikzcd}$$

the object-set of $|G\circ F|$ is $= P/\sim$.