Talk, Sept20

1. Commutative ring :

a set *R*;

operations:

plus = +:RxR--->R , a *binary* operation;
*times = *:RxR--->R*, another *binary* operation;
constant elements *zero = 0, one = 1* in *R* ;

laws:
unit laws – *zero* for *plus*; *one* for *times*;
associative laws (A+), (A*)
commutative laws (C+), (C*)
for plus and times
distributive law (D) : “times distributes over plus”.

The (A) and (D) laws are *ternary* in the sense that they use three independent variables. The © laws are *binary*. The unit laws are *unary.*

The concept of commutative ring (CR) is *equational*, defined by operations and identities. In category theory, we have a way of saying this: the category of CR's is finitary-monadic over Set, the category of sets.

I am going to define a more complicated, but also quite similar concept, that of *omega category*.

First,

2. Omega graphs and operations on them

An omega-graph X is a graded set

{*X*-sub-*n*}-sub-*n*-in **N**

(elements of *X*-sub-*n* are called *n-cells of X*)

together with functions

d -sub-*n* : *X*-sub-*n* -------> *X*-sub-(*n-1)* ,

c -sub-*n* : *X*-sub-*n* -------> *X*-sub-(*n-1)*, for every $n>0$,

called *domain* and *codomain* functions (sometimes also *source* and *target*, respectively),

and the following laws are required:

for every n>1,

see (*), T1 .

Notation: see T2, T3.

When the set X-sub-$(n+1)$ is empty – and therefore, necessarily X-sub-m is empty for all $m>=n+1$ –- we have an *n-graph*. In practice, most of the time we deal with n-graphs, with varying finite n . A 0-graph is just a set; a 1-graph is what is usually called in category theory a graph.

For convenience, I now stop using "omega" in "omega graph"; I just say "graph"; for the ordinary graph, I will say 1-graph.

The category of graphs, OmegaGraph, or OG for short, is the obvious category: objects are the graphs ("U-small" graphs, in the sense of being in a fixed but arbitrary Grothendieck universe U); morphisms are the obvious structure preserving maps:

see T4

An omega category will have an underlying omega-graph – just as a CR has an underlying set. Moreover, the concept of omega category will be equational – but now not over Set, but over OG. Operations will no longer have tuples (x,y,...) of elements as arguments , but rather *systems of elements indexed by finite graphs*; the "arities" are not (just) finite sets as in ordinary algebra, but (certain) finite graphs.

Given a(n omega) graph X, and a finite graph A, an "arity", a system of elements of X indexed by A is, naturally I hope, a graph-morphism a:A--->X, a morphism in OmegaGraph.

Example: In an ordinary category (which will, essentially, be an example of an omega category) we want to talk about composing two 1-cells f and g, with the proviso that c(f)=d(g)=y :

$$x \xrightarrow{f} y \xrightarrow{g} z ,$$

and we intend that the composite, denoted gf, should be a 1-cell with d(gf)=x , c(gf)=z :

$$x \xrightarrow{gf} z.$$

This can be expressed by saying that the composition operation *comp* is a function on the set of all morphisms <2>----->X to X-sub-1 , where <2> is the graph

$$[\ 0 \xrightarrow{3} 1 \xrightarrow{4} 2] .$$

with some commutative diagrams expressing the domain/codomain information on the composite.

Let us write [A,X] for

hom-sub-OG(A,X) ,

the set of all morphisms from the graph A. Also

note that X-sub-1 is, essentially, the same as
2
[<1>,X], where <1> is the graph [0--->1]. Thus, we will have that comp, the composition operation, will be a function of sets,

comp: [<2>,X] –------> [<1>,X] .

In fact, all the operations, even the composite ones (the analogs of *terms* in ordinary logic/algebra), in all the many different versions of strict, semistrict, and “weak” omega categories, will be of the form

PHI: [A,X] –-> [B,X]

on an (omega) graph X, with A, B *finite* graphs – in fact, very special finite graphs, the so-called *Batanin cells* (my name for them).

3. The main examples

An *n*-category, in a general sense of the term is a “category” structure on an *n*-graph. More particularly, it is a concept monadic over the category of *n*-graphs. This is not a definition yet. I first give the main examples.

We all know what a 1-category is: it is what is usually called a category. The main example of a 1-category is the category of sets, Set. There is just one problem with this: size. The more rigorous concept is category of U-small sets, Set-sub-U, with any Grothendieck universe U. However, we will usually suppress reference to U.

With the notion of category, the morphism of categories is an immediate notion: it is that of a functor, structure-preserving map from a category to another. Thus, we get the category Cat of (U-)small categories; to be sure, no longer U-small, but U-sub-1 small, for the next Grothendieck universe U-sub-1, for which U is an element of U-sub-1. However,   Cat has an additional structure, that of natural transformations: we have the concept of a natural transformation h:F--->G for two *parallel* functors F and G:

$$
\begin{tikzcd}
X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow h"] & A
\end{tikzcd}
$$

Thus, Cat becomes a *2-category*, “2-category” being an algebraic (monadic) concept over 2-Graph, the category of 2-graphs. 2-categories appear early in category theory; Mac Lane introduces them early in his book.

You will not be surprised to hear that the category of 2-categories (with the obvious, structure-preserving morphisms, is in fact, not only a 2-category, but a 3-category as well. And so on, for all finite n.

What about the main example for a “real” omega-category? I am sorry to say this, but this is the omega category of all small omega categories! No such thing, in a natural way, as an omega+1 category!

4. Strict omega categories

For the definition, see T5 to T13.

Summary: An omega category X has an underlying (omega) graph , also denoted by X. It has

unary operations

$$id_r : X_n \longrightarrow X_{n+1} ,$$

and binary operations

$$comp_{m,n} : X_{m,n} \longrightarrow X_r ,$$

for each pair (m,n) of positive integers, where $r=max(m,n)$, and the set $X_{m,n}$ consists of those pairs (a,b) where a is in X_m, b is in X_n, and

$$(c^k)(a) = (d^k)(b) ;$$

here, $k=min(m,n)-1$, and c^k is the "k-dimensional codomain", i.e., for a in X_m,

$$(c\^{}k)(a) \; = \; \underbrace{c\; c\; \ldots\; c}_{m\text{-}k \text{ times}}\; (a)$$

and similarly for *d^k*.

Briefly put, the composite *ab* is defined provided *a* and *b* “meet” at the level

$$k \; = \; min(dim(a),dim(b))\text{-}1,$$

meaning that the *k*-level codomain of *a* equals the *k*-level domain of *b* .

Note that we are using here a diagrammatic notation, whereby we write *fg* for what, usually in category theory with the functional notation, one would write as *gf*.

We have five categories of laws:

1) domain/codomain laws, regulating the domain/codomain of the identity and composite cells,

2) unit laws,
3) associative laws,
4) distributive laws
5) commutative laws.

The associative and distributive laws are ternary, meaning that each has three independent variables; the unit and the commutative laws are binary.

The associative and distributive laws have the usual forms:

$$(ab)e = a(be) ,$$
$$(ab)e = (ae)(be),$$
$$a(be) = (ab)(ae).$$

Although these formulas are completely precise, their simplicity is deceptive. In the second formula, a “distributive” law, the operation $(-)e$ distributing over the “addition” $a(+)b$, we see five compositions:

$$ab , \ (ab)e , \ ae , \ be, \ (ae)(be) .$$

The meanings of these compositions are determined by the dimensions of the variables *a*,*b*,*e*. With those dimensions given, the elements *a*,*b*,*e* also have to satisfy composability conditions for the composites to make sense.

The equalities are required to hold provided all expressions involved are well-defined; " ab is well-defined" means that *(a,b)* is in *X_m,n*, where *a* is in *X_m*, *b* is in *X_n*.

To be more precise, for instance in the case of associativity: there is one identity as an axiom for every triple *(m,n,p)* of positive integers such that :

whenever *a* is in *X_m*, *b* is in *X_n* and *e* is in **X_p**, then each of the four composites

ab, *be*, *(ab)e*, *a(be)*

are well-defined.

I say that when this is the case, the triple *(m,n,p)* of integers is *associative*. It turns out that this is the case if and only if two of the numbers m,n,p are equal, and the third is greater than or equal to they/them. If so, then, as a consequence, *(ab)e* and *a(be)* will be parallel cells of dimension *max(m,n)*; the axiom requires that they be equal.

It is very important to the story that we only require an equality of two terms when we already know that they denote elements *u* and *v* of some equal dimansion that are *parallel* to each other: *d(u)=d(v),* c*(u)=c(v)* .

The commutative law, although it is “only” binary, referring to two independent variables, it is a bit more complicated. For the formal statement, see T11 to T13. To a motivating discussion, we return after “Batanin cells”.

By *omega-category*, we will mean strict omega category thus defined. For a finite *n*, an *n*-category could be defined as an omega-category in which all *m*-cells, for all *m>=(n+1*), are identity cells. Essentially equivalently, an *n*-category is a structure with an underlying *n*-graph; with operations   id_r and comp_l,m as above, but for r<n and l,m<=n only; and with the laws as above involving the restricted set of operations.

5. Batanin cells

Batanin cells (my terminology), or B-cells for shorter, are for omega categories as multi-variable polynomials with integer coefficients for commutative rings.

Given a set x[n] ={x_1,...,x_n} of n distinct variables, the free CR on the set x is the ring of polynomials **Z**[x[n]] = **Z**[x_1,...,x_n], with **Z** the ring of integers, and with indeterminates x_1,...,x_n . I take **Z**[x[n]] to be the set of the usual normal form sum of non-zero monomials

Now, let X be an arbitrary set; denote by **Z**[X] the CR that is freely generated by X. Given any n , any map a:x[n] –--> X, and any polynomial f in **Z**[x[n]], we have f(a), an element of **Z**[X], “the value of f at a “, as F(a)(f) for

F : Set –-----> CR

the left adjoint to the forgetful functor CR--->Set (F([x[n]) = **Z**[x[n]] , F(X) = **Z**[X]). We have a simple fact: the mappings

PHI-n : **Z**[x[n]] x [x[n], X] -----> **Z**[X

(f in **Z**[x[n]], a in [x[n], X])
|------>f(a) in **Z**[X]

form a family <PHI-n >(n-in-**N**) that is almost (jointly) a bijection.

First of all, the PHI-n's are jointly surjective. Now, impose an arbitrary linear order on the set X, and on x[n], the natural order; allow only strict (1-1) order-preserving maps a:x[n] –--> X only, and allow only polynomials f in **Z**[x[n]] that contain all of the variables in x[n] at least once (in a monomial with a non-zero coeefficient). With these restrictions, the modified map PHI-n forms a jointly bijective family.

This statement is a way of saying – somewhat pedantically – that every element of $\mathbf{Z}[X]$ can be uniquely written in the usual normal form of a polynomial.