**PART 2**

M. Barr's theorem

Let: $C$ small regular category.

We have: $i : \text{Reg}[C, \text{Set}] \to [C, \text{Set}]$ (full inclusion)

$s = \text{Set}$

$\varepsilon_C : C \to [\text{Reg}[C, \text{Set}], \text{Set}]$

(definition)

(Theorem) $\text{ev}_C : C \to [\text{Reg}[C, \text{Set}], \text{Set}]$

is a regular full embedding.

(Similarly to "Joyal's theorem", $T_3$?)

The facts that the codomain is regular category, and that $\text{ev}_C$ is a regular functor are automatic. To be proved: $\text{ev}_C$ is full and faithful.
Duality: properties of the category of sets

In any category:

Definition: a QE (quotient of equivalence relation)

Diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow{g} & \downarrow{h} \\
& & Z
\end{array} \]

(*)

Such that \((f, g)\) is an equivalence relation(...)
and \(h\) is a coequalizer of \((f, g)\).

Fact: in Set, a (small) product \( \prod_{i \in I} \) of QE diagrams in QE again
(because (*) is QE in Set \( \iff \)
\((f, g)\) is a kernel-pair of \(h\),
and \(h\) is surjective)
Also: a filtered colimit of $\mathbf{QE}$ diagrams is a $\mathbf{QE}$ diagram in $\mathbf{Set}$.

Denote: $\mathbf{QE/FL} \mathbf{[C,S]}$: full sub-cat of $\mathbf{[C,S]}$ of the functors $\mathbf{C} \rightarrow \mathbf{S}$ that map $\mathbf{QE/FL}$ diagrams to $\mathbf{QE/FL}$ diagrams.

$\Pi/\mathbf{FC} \mathbf{[X,S]}$: functors $\mathbf{X} \rightarrow \mathbf{S}$ that map small-product diagrams in $\mathbf{X}$ to such in $\mathbf{S}$, and similarly for (small) filtered colimit diagrams.

For $\mathbf{S} = \mathbf{Set}$, and for any $\mathbf{X, C} \in \mathbf{CAT}$ we have a bijection

$$
\begin{align*}
\mathbf{C} \quad &\longrightarrow \quad \Pi/\mathbf{FC} \mathbf{[X,S]} \\
\Psi_{C,X} \quad &\quad \mathbf{X} \quad \longrightarrow \quad \mathbf{QE/FL} \mathbf{[C,S]}
\end{align*}
$$
defined by:

\[
\begin{array}{c}
\Sigma \xrightarrow{\pi} \Pi/\mathcal{FC} [X, S] \\
\xrightarrow{\psi G, X} \Psi G, X
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{P} Q/E/FL [G, S] \\
\xrightarrow{\psi G, X} \Psi G, X
\end{array}
\]

\[
\begin{array}{c}
\Sigma \xrightarrow{P} R \\
\xrightarrow{\psi G, X} \Psi G, X
\end{array}
\]

\[
\begin{array}{c}
\uparrow \quad \leftrightarrow \quad \text{incl.} \Sigma \xrightarrow{\text{incl.} P} \text{incl.} R
\end{array}
\]

the exponential adjunction for CAT, Set.

This is purely formal, by said duality properties of Set.
Let \( \text{QE/FL} \) denote the 2-category of all categories having the QE and FL operations defined on them, with 1-cells the QE/FL functors and 2-cells: all nat. transf's.

Similarly: \( \Pi/\text{FC} \).

We obtain, purely formally, the 2-adjunction

\[
\begin{array}{ccc}
\text{QE/FL}^\text{op} & \xleftarrow{(-)^\#} & \text{QE/FL}[-\to \text{Set}] \\
\downarrow \cong & & \downarrow \\
\Pi/\text{FC} & \xrightarrow{(-)^*} & \Pi/\text{FC}[-\to \text{Set}]
\end{array}
\]

Comm.: \( \text{C} \xrightarrow{E_C} \text{C} \xrightarrow{C^*\#} \)

Theorem (MM, 1988) for small Barr-exact category \( \text{C} \), \( E_\text{C} \) is an equivalence of categories.

NB: this \( E_\text{C} \) is like the one on T.3.6.2: the same function in fact, with codomain restricted.
The theorem says: in Barr’s theorem
the image of the full embedding
\[ \text{ev}_C : C \to \text{Reg} [C, \text{Set}], \text{Set} \]
(see: T3.8)
consists of the \( \Pi/\text{FC} \) preserving
functors \( \text{Reg} [C, \text{Set}] \to \text{Set} \).

**Definition**
A **dual-regular category** \( X \)
is one from which there is a small
regular category \( C \) such that
\[ X \cong \text{Reg} [C, \text{Set}] . \]

Let \( \text{Exact} \) be the 2-category of
small exact categories, and regular functors;
\( \text{Dual Reg} \) the full sub-2-category
of \( \Pi/\text{FC} \) on the dual regular categories.
We obtain a biequivalence \[ F = (-)^\# \]
\[ G = (-)^* \]

\[ \text{Exact} \]
\[ \text{Dual-Reg} \]

\[ \varepsilon : FG \rightarrow 1 \]
\[ \eta : 1 \rightarrow GF \]

**Corollary (John Bourke)**

The "limit theorem" holds for Dual Reg: any \( \text{[weighted]} \) pseudo-limit of dual regular categories, and \( \Pi/FC \) functors between them is again dual regular.

?: see later!
Pseudo-natural transformations (psnt's)

In what follows,

\[ X, \quad \lambda A, \quad \lambda M : \quad 2 \text{-categories} \]

0-cells: \( X, Y, \ldots \quad A, B, \ldots \quad M, N, \ldots \)

2-functor: \( X \xrightarrow{F} A : \) structure preserving
("on the nose", strictly)

psnt: \( X \xrightarrow{F} A : \)

\[ \varphi = (\langle \psi_X \rangle, \langle \psi_f \rangle : f : X \to Y) \quad \text{where} \]

\( X \in X_0 \): \( \psi_X : FX \to GX \) (in \( \lambda A_1 \), of course)

\[ \text{and} \quad \psi_f : (Gf) \circ \psi_X \xrightarrow{\cong} \psi_Y \circ (Ff) \quad (\in \lambda A_2) \]

\( X \xrightarrow{F} A \)

\[ F \downarrow \]

\( Ff \downarrow\]

\( G \downarrow \]

\( \psi_f \]

\( \psi_Y \]

\( X \xrightarrow{f} Y \)
Satisfying:

1) \( \begin{array}{c} \xymatrix{ & X \\ f \ar[r] & Y \ar[d] & g \ar[r] & X_2 \\ & Y } \end{array} \)

\[ \begin{array}{c} FX \xrightarrow{\varphi_X} GX \\ FF \xrightarrow{Fm} Fg \\ FY \xrightarrow{\varphi_Y} Gy \\ GF \xrightarrow{Gg} Gg \end{array} \]

Commutes:

\[ (Gm) \varphi_X = \varphi_Y (Fm) \]

2) \( \begin{array}{c} \xymatrix{ & X \\ f \ar[r] & Y \ar[d] & g \ar[r] & Z \ar[d] \\ & Y \ar[d] & \ar[r] & Z \ar[r] & Z } \end{array} \)

\[ \begin{array}{c} FX \xrightarrow{\varphi_X} GX \\ FF \xrightarrow{Ff} GF \\ FY \xrightarrow{\varphi_Y} GY \Theta \xymatrix{ F(gf) \ar[r] & G(gf) \\ \ar[r] & \ar[r] & G(gf) } \\ FZ \xrightarrow{\varphi_Z} GZ \\ FZ \xrightarrow{\varphi_Z} FZ \end{array} \]

\[ \varphi_{gf} = (Gg) \varphi_f \circ (\varphi_g (Ff)) \]

End of definition

31 psit of 2-functors (!)
Modification:

\[
\begin{array}{ccc}
F & \downarrow & \\ & \Lambda & \downarrow \\
\chi & \Psi & \chi \\
\end{array}
\]

\[\mu = \{ \Psi_x \}_{x \in X}, \mu : \Psi_x \rightarrow \Psi_x \in X^2\]

Satisfying

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
FX & \xrightarrow{Ff} & FY \\
\end{array}
\]

\[
\begin{array}{ccc}
\Psi_X & \xrightarrow{\mu_X} & \Psi_X \\
\downarrow & & \downarrow \\
\Psi_Y & \xrightarrow{\mu_Y} & \Psi_Y \\
\end{array}
\]

\[
\begin{array}{ccc}
\Psi_Y (FF) & \rightarrow & \Psi_Y (FF) \\
\mu_Y & \rightarrow & \mu_Y \\
\end{array}
\]

\[
\begin{array}{ccc}
(Gf) \Psi_X & \xrightarrow{\mu_X} & (Gf) \Psi_X \\
\Downarrow & & \Downarrow \\
\Psi_Y & \xrightarrow{\mu_Y} & \Psi_Y \\
\end{array}
\]
GRAY (after John Gray) is a 3-dimensional category, with the underlying 3-graph, with

- 0-cells: 2-categories
- 1-cells: 2-functors
- 2-cells: pretr's
- 3-cells: modifications

-- more on the def'n of GRAY: See later.

---

Why we need pretr's:

Recall limits in categories:

Let: \( C, G: \text{cat}'s, G\small; e.g. G = \begin{pmatrix} 1 & \downarrow 2 \\ \downarrow 3 & \downarrow 4 \end{pmatrix} \)

\( \{ \text{a } G\text{-type diagram } G \rightarrow C \text{ in } C \} \)

\( B \downarrow f \) in the example

\( C \rightarrow A \)
Definition \( f : \) a cone on \( \Gamma \) with vertex \( X \):

\[
\begin{align*}
\{ f : \Gamma X^\Gamma & \to \Gamma, \\
G & \xrightarrow{\Gamma X^G} C \}
\end{align*}
\]

natural transf

where \( \Gamma X^\Gamma \) is the constant-\( X \) functor.

Example: when \( G \) is as above, \( \Gamma \) and \( f \) look like this:

\[
\begin{array}{c}
\xymatrix{ 
X \ar[d]^{s_1} \ar[d]_{s_2} \ar[dr]_{s_0} & \ar[d]^{s_2} \\
\Gamma X^\Gamma \ar[r]_{s_0} & C \ar[r]_g & \ar[r]_g A}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{ 
X \ar[r]^{s_1} & B \\
\ar[ru] \ar[ru]_s \\
C \ar[r]_g & \ar[r]_g A}
\end{array}
\]

`Pullback-shaped diagram`

\( s_0 \) is then uniquely recovered.
For fixed $P$ and $X$, the set of cones on $P$ with vertex $-X$,

$$\text{Cone}_P(X) \overset{\text{def}}{=} \text{Nat}(\mathbb{F}X^\uparrow, P) \quad \text{(a set)}.$$ 

For fixed $P$, but variable $X$ in $C$, we have the functor:

$$\text{Cone}_P : C^{\text{op}} \rightarrow \text{Set}$$

\[
\begin{align*}
\begin{array}{ccc}
X & \mapsto \text{Cone}_P(X) & \varphi : X^\uparrow \rightarrow P \\
\downarrow f & & \downarrow f^* \\
Y & \mapsto \text{Cone}_P(Y) & \varphi^* : Y^\uparrow \rightarrow X^\uparrow \rightarrow P
\end{array}
\end{align*}
\]

Definition: $P$ has a limit in $C$ if $\text{Cone}_P : C^{\text{op}} \rightarrow \text{Set}$ is representable.
There is $(L, \pi)$

$L \in \mathcal{C}$, $\pi \in \text{Cone}_p(L)$

such that the natural transformation

$$
\theta_{u,u} : \mathcal{C}(\frac{\pi}{\pi}, L) \rightarrow \text{Cone}_p
$$

$$(\theta_{u,u})_X : \mathcal{C}(X, L) \rightarrow \text{Cone}_p(X) \quad (X \in \mathcal{C})$$

$$X \xrightarrow{f} L \quad \rightarrow \quad \pi \circ f^*$$

is an isomorphism (as a nat. transf., equivalently, for each $X$ separately).

$L$ is the limit, $\pi$ is the limit cone.

Returning to the example: $G = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 \end{pmatrix}$, and $p \mathcal{T}_3 10.3$
There is a unique \( x \in L \) such that

\[
L \ni x \quad \text{for all} \quad X \ni x
\]

is a limit cone iff

\[
\exists \gamma, \quad L \ni \gamma \quad \text{and} \quad L \ni \gamma
\]

\[
\quad \text{i.e.,} \quad T_3
\]
CAT as a category has all small

limit and colimits (as a locally

finitely presentable category) — the limits

computed in an expected manner (simply).

(colimits: not simple at all!)

With

\[
\begin{array}{ccc}
I B & \xrightarrow{F} & I A \\
\downarrow & & \\
C & \xrightarrow{G} & I A
\end{array}
\]

Categories and functors, the pull back.

\(P\) has: \(P_0 = \{(B, C) : F(B) = F(C)\}\)

\(P_1 = \ldots \) as expected.

However: with \(A, I B, C\) having terminal

objects, \(F, G\) preserving them (taking a

terminal to a terminal) — it does not follow

that \(P\) has a terminal object: for \(1_B, 1_C\)

the terminals in \(I B, C\), \(F(1_B)\) and \(F(1_C)\)

are terminals in \(I A\), but may not be equal.

(Explicit example is easy.)
Thus, the \(2\)-cats \(\text{Reg, Exact, Dual Reg, ...}\) do not have ordinary pullbacks — however, they have small \underline{pseudo-limits} and \underline{pseudo-colimits}.

\[
\begin{align*}
\text{\{ Categorification of the notion of limit \}}
\end{align*}
\]

(Max Kelly and Ross Street):

Let: \(G: 2\)-category

\(\Gamma: G \to C: 2\)-functor

\(W: G \to \text{Cat}: 2\)-functor ("weight")

(special case: \(W = \Gamma^1\), constant \(1\) (terminal cat) functor, with \(1\) the terminal 2-cat).

\text{Definition}\ \exists \text{ is a } W\text{-weighted pseudo-cone on } \Gamma \text{ with vertex } X \text{ if }

\[
\exists: W \xrightarrow{\text{pseudo-nat}} C(X, \Gamma(-))
\]
\[ T_3 \] 10.10

\[ G \xrightarrow{bsnt} W \xrightarrow{\delta} CAT \]
\[ C(X, \Gamma(-)) \]

Where

\[ G \xrightarrow{W} C \xrightarrow{C(X,-)} CAT \]

A representable 2-functor

\[ Y \xrightarrow{1} C(X_Y) \]

A category

When \( W = \Gamma^1 \), above reduces to

\[ G \xrightarrow{\text{part}} W \xrightarrow{\delta} \Gamma \xrightarrow{L^1} C \]

The category \( C_{\Gamma^W(X)} \) has objects these \( \delta \)'s, arrows: modifications,
Cone $^W_p(X)$ as a function of $X \in C$

in a 2-functor:

$$\text{Cone}^W_p : C^{op} \longrightarrow \text{CAT}$$

\[
\begin{array}{c}
\xymatrix{
X 
\ar[r] & 
\text{Cone}^W_p(X) 
\ar[d]^{f^*} 
\ar[r] & 
W 
\ar[r]^f & 
C(X, p(-)) \\
Y 
\ar[u]^f 
\ar[r] & 
\text{Cone}^W_p(Y) 
\ar[r] & 
W 
\ar[r] & 
C(Y, p(-)) 
\ar[l]_{(-) \circ f} \\
& 

}\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
X 
\ar[r] & 
\text{etc} \\
Y 
\ar[u]^f 
\ar[r]^g & 

}\end{array}
\]

Definition: For $\Gamma : C \to \text{Cat}$, $\Gamma$ has $W$-weighted pseudo limit in $C$ if $\text{Cone}^W_p : C^{op} \to \text{CAT}$ is (up-to-iso) representable.
Examples:

1) Let: \( G = \begin{array}{ccc}
1 \\
\downarrow 3 \\
2 \\
\downarrow 0 \\
\end{array} \)

and \( W = \{ 1 \} \); the limit: pseudo pullback;

exists in \( \text{CAT} \): the pseudo-pullback of

\[
\begin{array}{ccc}
B & \xrightarrow{F} & A \\
\downarrow & & \downarrow \\
C & \xrightarrow{G} & A \\
\end{array}
\]

\( P \rightarrow B \)

\[
\begin{array}{ccc}
P & \xrightarrow{\varphi} & B \\
\downarrow & \swarrow & \downarrow F \\
C & \xrightarrow{\varphi'} & A \\
\end{array}
\]

(slight cheating)

is given by:

\[ P_0 = \{ (B, C, \varphi) : F(B) \xrightarrow{\varphi} G(C) \} \]

\[ P_1 : (B, C, \varphi) \rightarrow (B', C', \varphi') \, \text{is: any} \]

\((b, c) = (b : B \rightarrow B', c : C \rightarrow C')\)
such that

\[ F(B) \xrightarrow{\psi} G(C) \]

\[ F(B') \xrightarrow{\psi'} G(C') \]

Reg, Exact, Dual Reg, ... have pseudo pullbacks computed as in CAT

for Dual Reg: not obvious!

(John Bourke's "corollary" above)

but for Reg, Exact, ... it is obvious' (easy).

2) Let: \( G = \) the terminal 2-cat \( I \)

(one object, one arrow, one 2-cell)

with a small cat \( T \); \( W = T^T : I \to \text{CAT} \)
\[ P = \{X\} : T \rightarrow \text{C} \quad (X \in \text{C}_0) \]

The \(W\)-limit of \(P\) is called the cotensor

\[ T \otimes X \quad \text{(if it exists)} \]

In \(\text{C} = \text{CAT}\), (small) cotensors exist:

\[ T \otimes X = [T, X] \]

\(\text{Reg}, \text{Exact}, \ldots, \text{Dual Reg}\) inherit cotensors from \(\text{CAT}\)

(for \(\text{Dual Reg}\) : not obvious)