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Coherent categories and Godel completeness

1. We give the definition of “coherent category”, and give a formulation and a proof of “Godel completeness”. These concepts and mathematical facts are not new by any means. The raison d'etre of the present exposition is the desire to bring out the fact that the proof as given here of Godel completeness is direct: it proceeds from the definition of “coherent category” directly, without intervening lemmas asserting derived properties of a coherent category --- which fact I consider to be a good sign that the definition of “coherent category” is a “good” one: it is an economical one, at least.

Let us point out at the start that **Set,** the category of sets, is a coherent category. In fact, the concept of “coherent category” is motivated by the desire to have a general concept of a category resembling Set in all essential “coherent” properties – where “coherent property” is something quite restrictive, but still sufficient for a large part of first-order logic.

To illustrate the economy of our proof of completeness, let me point out that it is true (and easy to prove!) that, in any coherent category, the composite of two extremal epimorphisms (e.e.'s) is again an e.e. This fact *may* play the role of a lemma in a proof of “Godel completeness” (in fact, it does so in the proof a version of Godel completeness based on Grothendieck topologies (see my paper […]), but it does not figure in the proof to be given here.

2. On the other hand, Godel completeness can be used to prove properties of coherent categories like the one just mentioned. We have in mind the so-called exactness properties, widely used in category theory. I will give a definition of “exactness property” *for* a coherent category; I will call them *coherent exactness properties*.

A coherent exactness property (c.e.p.) P(**C**) is either possessed or is not possessed by any given coherent category **C**. For instance, the property P(**C**): “in **C** , the composite of any two e.e. morphisms is e.e.” turns out to be a c.e.p. “by its very form”, one that happens to hold true in all coherent categories. On the other hand, the c.e.p. that “every morphism is a monomorphism” is true only in a very restricted class of coherent categories, the distributive lattices. The property “every e.e. morphism has a section (left inverse)” is *not* a coherent exactness property:

firstly, it is not formulated as such to conform to our future definition of "coherent exactness property", but also, it *cannot be* formulated as such, as we will *prove* below.

The main result to look out for is the fact that

any coherent exactness property possessed by Set, the category of sets, is shared by all coherent categories.

This will be seen to be a direct consequence of what we will call and prove as "Godel completeness for coherent categories".

For instance: in Set, the extremal epimorphisms are seen to be the same as the surjective functions; and one sees (or "knows") that the composite of surjective functions is again surjective. It follows that the composite of e.e. morphisms is e.e., in any coherent category.

Since the last-mentioned fact is easy to prove anyway, completeness may not appear to be an important tool. Its importance is enhanced, however, by the number of applications of the same kind, some of which will be shown in this exposition.

3. The category theorist will see that what is done here is quite familiar in the context of (projective) limits in categories, and also, in 2-dimensional limits in 2-categories. A diagram D in a category **C** of the shape of a limit diagram (for instance, the diagram

[1] [diag missing]

which is the shape of a pullback; formally, it is a map D:**Pb --->C**, where **Pb** is the graph

[2] [diag missing])

is in fact a limit diagram if and only if the composite, shown in the case of te pullback:

[3] [diag missing]

is a limit diagram in the category Set. This fact is almost tautological; it should be seen directly from the definition of "limit", without referring to things like Yoneda lemma and the like. The upshot of this is that an exactness property for limits can be shown to hold *in allcategories* by verifying it in Set. Here is an example: consider the following diagrams with unnamed objects and arrows

[4] [diag mising]

where we assume that both the squares 1 and 2 are commutative, and 3 is the obvious “composite” of 1 and 2. The assertion is that if from the three diagrams, 1 and 2 are pullbacks, then so is 3; and if 2 and 3 are pullbacks, then so is 1. I am saying that it suffices to show these facts to be true in Set, to conclude that they are true in *any* category. Indeed: for instance, for the first assertion, assume that 1 and 2 are pullbacks. Let X be any object in our category C, to show that 3 C(X,3) is a pullback in Set – which will be enough by what we said above about limit diagrams in general. We do have (by the “only if” part of our general statement) that C(X,1) and C(X,2) are pullbacks. Since C(X.--) is a functor, C(X,3) is a the composite of C(X,1) and C(X,2) in the same sense as 3 is the composite of 1 and 2 (the long horizontals are composites of short ones). Therefore, C(X,3) is a pullback, since said property is accepted to hold in Set (although I have not checked it here) – and that was the thing to be proved as stated above.

We can – and we will, despite the expected dismissal of the effort as superfluous – formulate precisely what we call an exactness property for finite limits, and we will have the – indeed trivial – theorem that the finite-limit exactness property (f.l.e.p.) P holds is *any* category C if and only if P holds in the category of sets. Every f.l.e.p. is at the same time a c.e.p.; thus, *for coherent categories*, the “trivial” completeness result for f.l.e.p.'s is a consequence of “Godel completeness”. But this is an opportunity to point out an important difference between the coherent and the finite-limit frameworks. As a matter of fact, it is meaningful to ask if a given coherent exactness property (c.e.p.) does or does not hold in any given category, similarly to asking the similar question in case of an f.l.e.p. However, now the completeness does not hold: unlike as in the limit situation, it is not enough to check the property holding in Set, to conclude that it holds in *any* category. We will see why this is the case when we come to it –- but it is good to realize from the start that the coherent framework is more sophisticated than the finite-limit framework.

[to be continued]