$\boxed{1}$ Good diagrams

(I am using notation matching that of the set "The κ-good small object argument".)

(1.1) Let $(P, <)$ be a partially ordered set with a bottom (least) element $\perp$ ($\perp \leq x$ for all $x \in P$), and assume that $(P, <)$ is well-founded: there is no infinite descending sequence of elements of P:

$$\nexists \; (x_0 > x_1 > \cdots > x_n > \cdots \quad \text{for all } n \in \mathbb{N})$$

$(x_n \in P)$. As a consequence, appropriate forms of transfinite induction and transfinite recursion are applicable, for proving assertions of the form "for all $x \in P$, $\Phi(x)$ holds", and for defining a function $\Phi : P \longrightarrow U$ (any set U) by specifying $\Phi(x)$ $(x \in P)$ in terms of the restriction of Φ itself to arguments $y < x$, i.e. the function

$$\Phi^{(x)} \stackrel{\text{def}}{=} \Phi \restriction \{y : y < x\} : \{y : y < x\} \longrightarrow U$$

for which $\Phi^{(x)}(y) = \Phi(y)$. Thus, if we have the 'functional equation'

$$\Phi(x) = \underset{\sim}{C}(\Phi^{(x)}) \qquad (*)$$

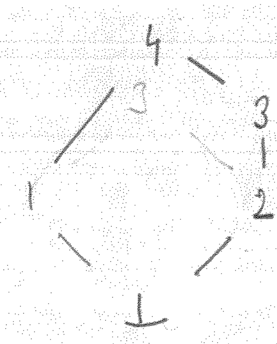
with any given function $\underset{\sim}{C}$ that assigns an element of U to any function of the form $Q \to U$ with $Q \subseteq P$, then there is a unique $\Phi : P \to U$ satisfying $(*)$.

I call $x \in P$ isolated if there is an element x^-, called the (immediate) predecessor of x, such that

$$y < x \iff y \leq x^- \qquad (y \in P).$$

All $x \in P$ that are not $\perp$, and not isolated are called limit (points (of P)).

Note: All finite posets with a bottom element qualify as our $(P, <)$. In the poset



1, 2, 3 are isolated, 4 is limit.

All non-empty well-ordered sets, in particular, all $\neq 0$ ordinals, also qualify. The isolated points are the same as the successor ordinals (in the set). The phenomenon of a limit point as 4 in the above example does not appear in the well-ordered case; all limit points x are such that $\{y : y < x\}$ is, in particular, infinite.

Let us refer to $(P, <)$ as specified as a good poset (although we have said very little so far by saying that something is 'good').

Let $\boxed{\kappa \text{ be a regular cardinal}}$ number; κ will be kept fixed in most of what we do.

Our set theory defines the concept so that we have: κ is an infinite ordinal number (hence, κ is equal to the set $\{\alpha : \alpha$ is an ordinal and $\alpha < \kappa\}$) such that every time we have: $\alpha < \kappa$ and a function $\langle \gamma_\beta \rangle_{\beta<\alpha}$:

$$\beta < \alpha \longmapsto \gamma_\beta < \kappa,$$

we also have: $\boxed{\sup_{\beta<\alpha} \gamma_\beta < \kappa.}$

$\left(\sup_{\beta<\alpha} \gamma_\beta = \bigcup_{\beta<\alpha} \gamma_\beta = \right.$ least ordinal δ such that $\gamma_\beta \leq \delta$ for all $\left.\beta < \alpha\right)$.

Definition of (infinite) regular cardinal (I)

An equivalent definition is: κ is an infinite cardinal number (an ordinal such that the cardinality of every $\alpha<\kappa$, $\#\alpha$, is less than κ) such that

$$\#X_\beta < \kappa \text{ for all } \beta<\alpha<\kappa \Rightarrow \#\Big(\bigcup_{\beta<\alpha} X_\beta\Big) < \kappa.$$

$\kappa = \aleph_0 = \omega$ is the least infinite cardinal number; it is regular; for our immediate goals, the most important example.

But $\aleph_1$ = the first uncountable ($> \aleph_0$) cardinal is regular too — and many others ...

$(P, <)$ is a $\underline{\kappa}$-good poset if it is good, and for every $x \in P$,

$$x\downarrow \underset{\text{def}}{=} \{y \in P : y \leq x\},$$

the set of predecessors of x, or, the down-set of x, is of cardinality $< \kappa$.

When $\kappa = \aleph_0$, this means that $x\downarrow$ is a finite set. Note that this condition alone ensures that P is well-founded! Of course, when $\kappa > \aleph_0$, well-foundedness is not implied by the 'κ-goodness' condition.

(1.2) $\mathbb{A}$ is a fixed category.

A diagram $A : P \longrightarrow \mathbb{A}$ $\quad (P = (P, \leq))$

(in notation: $A(x) = A_x$ $\quad (x \in P)$

$$A(x \leq y) = a_{xy} \quad (x \leq y \text{ in } P))$$

is *good* if $\overbrace{(P, <)}^{\text{the poset}}$ is good, and

for every limit point x of P, the object A_x is the colimit of the restriction of the diagram A to the subset

$$x{\downarrow\downarrow} \underset{\text{def}}{=} \{y \in P : y < x\};$$

or more precisely, the family $\gamma_x = \langle a_{yx} : A_y \to A_x \rangle_{y<x}$ is a *colimit* cocone on the diagram $A \restriction (x{\downarrow\downarrow})$. (The fact that γ_x is a *cocone* is already ensured.). The diagram $A : P \longrightarrow \mathbb{A}$ is *κ-good* if P is a κ-good

poset, and A is a good diagram.

Let us denote the colimit of $A : P \to \mathbb{A}$ by $A_\top$. If P has a top (maximum) element, denoted $\top$, then $A_\top$ can be taken (of course) to be colim A. We write, in general,

$$a_{x\top} : A_x \longrightarrow A_\top \qquad (x \in P)$$

for the colimit coprojections.

The <u>composite</u> of a good diagram $A : P \to \mathbb{A}$ is the coprojection $a_{\perp\top} : A_\perp \longrightarrow A_\top$. $\widetilde{\mathrm{Aboi}}\langle A\rangle \underset{\text{def}}{=} a_{\perp\top}$.

The composite of a good diagram may be regarded as a "transfinite composite of its <u>links</u>", where by a <u>link</u> of $A : P \to \mathbb{A}$ I mean any of the arrows

$$a_{x^- x} : A_{x^-} \longrightarrow A_x$$

for isolated points $x \in P$. When one constructs a good diagram $A : P \longrightarrow \mathbb{A}$ on a given good poset by recursion according to the well-founded order on P, the value $A_\perp \in \mathrm{Ob}(\mathbb{A})$ is "freely" chosen; ~~and~~ at each isolated $x \in P$, once we have A_{x^-}, the <u>link</u> $a_{x^-x} : A_{x^-} \longrightarrow A_x$ is "freely chosen"; however, for x limit, A_x and all the arrows $a_{yx} : A_y \to A_x$ $(y < x)$ are determined, at least up to isomorphism, by the requirement to form a colimit diagram.

1.3 Given a class $\underset{\sim}{C}$ of arrows in the category $\mathbb{A}$, a <u>transfinite composite of arrows in</u> $\underset{\sim}{C}$ is any arrow $f : A \to B$ that can be obtained in the form

$$\begin{array}{ccccc} a_{\perp\top} & : & A_\perp & \longrightarrow & A_\top \\ \| & & \| & & \| \\ f & & A & \longrightarrow & B \end{array}$$

for a good diagram $A : P \longrightarrow \mathbb{A}$ all whose <u>links</u> are in $\underset{\sim}{C}$. It turns out that <u>totally</u> ordered good diagrams are "sufficient" (and "transfinite composite" in the literature is meant in this well-ordered sense); although we need pushouts of the given links to get the links of the well-ordered substitute composite. For a precise statement, see 2. Proposition, in §3, p. 15 in "Rearranging ...". So: why the generalization to posets? The answer will be the application we will make to the General Completeness Theorem (GCT).

I now define the notation used in the Theorem of the set "The κ-good small object argument".

⊙ For a class $I \subseteq \mathrm{Arr}(\mathbb{A})$, $\boxed{Po(I)}$ is the class of pushouts of elements of I: $Po(I)$ is the class of all arrows $A \xrightarrow{f} B$ for which there exists a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ U \arrow[u] \arrow[r, "r"'] \arrow[ur, phantom, "\llcorner"] & V \arrow[u] \end{tikzcd}$$

where $r \in I$.

Cell For a class $J \subseteq \mathrm{Arr}(\mathbb{A})$, $\boxed{\mathrm{Cell}(J)}$ is the class of all transfinite composites of elements of J, 'transfinite composite' in the traditional well-ordered sense. $f \in \mathrm{Cell}(J)$ if_{def} there exist a well-ordered (totally ordered and well-founded) P and a good diagram $A : P \to \mathbb{A}$ such that $f = \langle A \rangle = a_{\perp\top} : A_{\perp} \to A_{\top}$.

notation

In the complicated notation "$\mathrm{Gd}_{\kappa}\,\mathrm{dir}_{\kappa}$" in loc. cit., we are referring to κ-good diagrams that are also κ-directed. A poset P is $\underline{\kappa\text{-directed}}$ (should be called "$<\kappa$-directed") if for all subsets X of P of cardinality $<\kappa$, there is $x \in P$ such that $X \leq x$ in the sense that $y \leq x$ for all $y \in X$. When $\kappa = \aleph_0$, we have the

the usual notion of 'directed'.

- For a class $J \subseteq \mathrm{Arr}(\mathbb{A})$,

$$\boxed{\underline{\mathrm{Gd}_\kappa \, \mathrm{dir}_\kappa \, (J)}}$$

is the class of all $f \in \mathrm{Arr}(\mathbb{A})$ for which

there exists a

κ-good and κ-directed diagram

$$A : P \longrightarrow \mathbb{A}$$

all whose links are in J

such that

$$f = \langle A \rangle = a_{\perp\top} : A_\perp \to A_\top$$

(*)

- $\mathbb{A}_\kappa$ denotes the class of <u>κ-presentable</u> objects in $\mathbb{A}$. See the definition of 'finitely presentable' in the set 2015 April 07 Sketch Notes. PDF, p S $\boxed{26}$, and instead of a directed diagram $A : I \to \mathbb{S}$, take a κ-directed one; you will then obtain

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the definition of 'κ-presentable' (should be: "<κ-presentable")

<u>Exercise</u>. In the category $\mathbb{A} = \mathrm{Set}$, the $\aleph_1$-presentable objects ('countably presentable') are the countable (possibly finite) sets.

① The phrase "h has the right lifting property with respect to I" means, in the terminology I used in class, that $f \perp h$ holds for all $f \in I$. Here, "$f \perp h$", or "f is left orthogonal to h", equivalently, "h is right orthogonal to f", means that every time we have

$$\begin{array}{ccc} U & \xrightarrow{u} & A \\ {\scriptstyle f}\downarrow & \circleddash & \downarrow{\scriptstyle h} \\ V & \xrightarrow[v]{} & B \end{array}$$

commutative, we have at least one

$d: V \to A$ such that

$$df = u \quad \& \quad hd = v:$$

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "f"'] & A \arrow[d, "h"] \\
V \arrow[r, "v"'] \arrow[ur, dashed, "d"] & B .
\end{tikzcd}$$

I have explained all the terminology for the Theorem on p. 1 of "The κ-good small object argument". Let me re-state the theorem here.

We take $\mathbb{A}$ to be a small-cocomplete category, κ an (infinite) regular cardinal, I a <u>small set</u> of arrows in $\mathbb{A}$ such that the domain of every arrow in I is κ-presentable.

(<u>Remark</u>: in the application we will have, the GCT, $\mathbb{A}$ will be locally finitely presentable, $\kappa = \aleph_0$, and every arrow in I will have <u>both</u> its domain and codomain a finitely presentable object).

The assertion is that any $f : A \to B$ in $\mathbb{A}$ can be factored as $f = h \circ g$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & \Theta & B \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

Such that

(i) $g \in \mathrm{Gd}_\kappa \mathrm{dir}_\kappa(\mathrm{Po}(I))$, i.e.,

there exists a κ-good κ-directed diagram $A : P \to \mathbb{A}$ all whose links are <u>pushouts</u> of arrows in I such that $g = a_{\bot\top} : A_\bot \to A_\top$;

and

$$\text{(ii)} \quad I \perp h \underset{\text{def}}{\Longleftrightarrow} i \perp h \text{ for all } i \in I.$$

(g is also seen to belong to $\mathrm{Cell}(\mathrm{Po}(I))$, but this is not important now).

(TO BE CONTINUED)