$\boxed{\text{SKETCHES}}$ April 7, 2015

Contents

SKETCHES

$\boxed{1}$ The category of sketches using a given set of specifications.

$\boxed{1.}$ Let: $\mathbb{G}$: a category

Example: $\mathbb{G} = \mathcal{G}raph$
the category of small graphs

$\mathbb{K}$: a set;
an element $K \in \mathbb{K}$ is called a <u>specification</u> <u>name</u>

$\mathbb{K} = \{K_{id}, K_{comm}\} = \mathbb{K}_{cat}$

some distinct symbols

a function

$$\varphi : \mathbb{K} \longrightarrow Ob(\mathbb{G})$$
$$K \longmapsto \varphi(K) = \overline{K}$$

also written

$\overline{K}$ is the K-<u>template</u>

$$K_{cat} : \{K_{id}, K_{comm}\} \xrightarrow{\overline{(-)}} Ob(\mathcal{G}raph)$$

$$\overline{K_{id}} = \boxed{\begin{tikzcd} 0 \arrow[loop above, "1"] \end{tikzcd}}$$

$$\overline{K_{comm}} = \boxed{\begin{tikzcd} & 1 \arrow[dr, "4"] & \\ 0 \arrow[ur, "3"] \arrow[rr, "5"'] & & 2 \end{tikzcd}}$$

A $(\mathbb{G}, \mathbb{K}, \varphi)$-sketch – or simply, a $\boxed{\text{sketch}}$, is:

$$S = (|S|, K(S))_{K \in \mathbb{K}} \qquad \text{where } |S| \in Ob(\mathbb{G}),$$

and

$$K(S) \subseteq \hom_{\mathbb{G}}(\overline{K}, |S|).$$

Example for the example:

Let $\mathbb{C}$ be a (small) category. Define a $\boxed{(\text{Graph}, \mathbb{K}_{cat}, \varphi_{cat})\text{-sketch}}$ $S = \boxed{Sk(\mathbb{C})}$ by:

$$|S| \underset{def}{=} |\mathbb{C}| = \text{the underlying graph of } \mathbb{C};$$

$$K_{id}(S) \underset{def}{=} \{ \sigma : \bar{K}_{id} \to |\mathbb{C}| \mid \sigma(1) = id_{\sigma(0)} \}$$

$$K_{comm}(S) \underset{def}{=} \{ \sigma : \bar{K}_{comm} \to |\mathbb{C}| \mid$$

$\sigma(5) = \sigma(4) \circ \sigma(3)$ in $\mathbb{C}$

Equivalently:

$$\begin{tikzcd} & \sigma(1) \arrow[dr, "\sigma(4)"] & \\ \sigma(0) \arrow[ur, "\sigma(3)"] \arrow[rr, bend right, "\sigma(5)"'] \arrow[rr, phantom, "="] & & \sigma(2) \end{tikzcd}$$

in the category $\mathbb{C}$ $\}$

A $\boxed{\text{morphism of sketches}}$ [$\mathbb{G}$, $\mathbb{K}$, φ : fixed]

$$F : S \longrightarrow T$$

is a morphism $F : |S| \longrightarrow |T|$ in $\mathbb{G}$

such that, for $\sigma \in K(S)$, we have $F \circ \sigma \in K(T)$:

$$\begin{tikzcd}
\bar{K} \arrow[r, "\sigma"] \arrow[rr, bend right, "F\circ\sigma"'] & {|S|} \arrow[r, "F"] & {|T|}
\end{tikzcd}$$

$$\begin{tikzcd}
\hom_{\mathbb{G}}(\bar{K}, |S|) \arrow[rr, "F\circ(-)"] & & \hom_{\mathbb{G}}(\bar{K}, |T|) \\
& = & \\
K(S) \arrow[uu, hook] \arrow[rr, dashed] & & K(T) \arrow[uu, hook]
\end{tikzcd}$$

Sketches and their morphisms form a category

$$\boxed{Sk(\mathbb{G}, \mathbb{K}, \varphi)}$$

<u>Example continued</u>: now, we have the category of <u>ca</u>tegory-<u>sketches</u>, $\boxed{Sk.Cat} = Sk(Graph, \mathbb{K}_{cat}, \kappa_{cat})$.

We have ~~(an)~~ full & faithful functor, an <u>inclusion</u> (- injective on objects -):

$$\begin{array}{lccc}
sk : & Cat & \longrightarrow & SkCat \\
& \mathbb{C} & \longmapsto & sk(\mathbb{C}) \\
& F\downarrow & \longmapsto & \downarrow sk(F) \\
& \mathbb{D} & \longmapsto & sk(\mathbb{D})
\end{array}$$

$\boxed{2.}$ Sketch semantics

Definition Let $\mathbb{S}$ be any category, $\alpha : A \to B$ an arrow of $\mathbb{S}$, S an object of $\mathbb{S}$. I write

$$\boxed{S \models \alpha}$$

and say: S satisfies the sketch <u>entailment</u> α ('if A, then B')

if

for all $f : A \to S$, there is (at least one)

$g : B \to S$

such that $g\alpha = f$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "f"'] & B \arrow[d, dashed, "g"] \\ & S \end{tikzcd}$$

For a set $\mathbb{R} \subseteq \mathrm{Arr}(\mathbb{S})$,

$$\boxed{S \models \mathbb{R}}$$

means $S \models \alpha$ for all $\alpha \in \mathbb{R}$.

<u>Definition</u> (logical consequence). For $\mathbb{R} \subseteq \mathrm{Arr}(\mathbb{S})$, $\varphi \in \mathrm{Arr}(\mathbb{S})$ we say: φ is a <u>consequence</u> of $\mathbb{R}$, and write

$$\boxed{\mathbb{R} \models \varphi}$$

if for all $S \in \mathrm{Ob}(\mathbb{S})$, $S \models \mathbb{R}$ implies $S \models \varphi$

Example of $\boxed{\text{SkCat}}$ (p. 3) continued:

We let: $\mathbb{S}$ of page 4 be

$$\mathbb{S} = \text{SkCat}$$

We define several morphisms of SkCat, $Ax_0, Ax_1, \dots$ and examine their meaning, i.e. the meaning of $S \models Ax_0, Ax_1, \dots$. We call arrows of a sketch-category such as SkCat (sketch-) entailments. A sketch entailment $\alpha: A \to B$ is read roughly as

"if A then B"

— hence the expression entailment. The sketch entailments $Ax_0, Ax_1, \dots Ax_5$ are intended to specify the concept of the notion of "category". Precisely speaking: consider the inclusion

$$sk: \text{Cat} \hookrightarrow \text{SkCat}$$

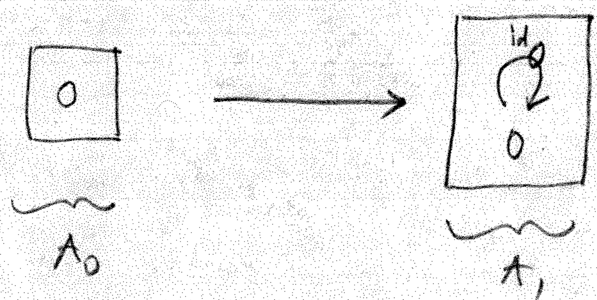
(see p $\boxed{3}$). We will have that the image of the functor sk, i.e., the category Cat itself, essentially, is given precisely by the set $\mathbb{R}_{\text{cat}} \underset{\text{def}}{=} \{Ax_0, \dots, Ax_6\}$; for $S \in \text{SkCat}$

$$\boxed{S \text{ is a category} \iff S \models \mathbb{R}_{\text{cat}}}$$ (see: box, p. $\boxed{4}$)

$\boxed{Ax_0 ::}$

(Ax_0 = Ax_{idex} = Axiom for the existence of the identity arrows)

Ax_0 :



explanation of shorthand

$A_0 \in SkCat$; $|A_0| \in Graph$;

$$Ob(|A_0|) = \{0\}$$

$$Arr(|A_0|) = \emptyset$$

$$K_{id}(A_0) = K_{comm}(A_0) = \emptyset$$

[later, when a specification set is not mentioned, it is to be understood that it is empty]

A_1 :: $Ob(|A_1|) = \{0\}$, $Arr(|A_1|) = \{1\}$ $\Big\}$ $|A_1| = |\overline{K}_{id}|$

$$K_{id}(|A_1) = \{id_{\overline{K}_{id}}\} = \{i\}$$

abbreviated (for below)

Ax_0 is the unique map $A_0 \to A_1$ in SkCat:

$$0 \longmapsto 0$$

Note: $sk(\mathbb{C}) \models Ax_0$ ::

see: p. 3 bottom

$$\begin{tikzcd}
\boxed{0} \arrow[r, "Ax_0"] \arrow[dr] & \boxed{0 \circlearrowleft id_0} \arrow[d, dotted, "\psi"] \\
 & {|\mathbb{C}|}
\end{tikzcd}$$

($\forall$ (given any:)) $\varphi :=: \ulcorner U \urcorner$

$\ulcorner U \urcorner(0) = U \in Ob(\mathbb{C})$

$\exists\,(!,\text{now})$;

because:

commutativity requires:

$$\psi(0) = U$$

$\psi : A_1 \to sk(\mathbb{C})$ sketch map:

$$\begin{tikzcd}
K_{id} \arrow[r, "i'"'] \arrow[rr, bend right] & {|A_1|} \arrow[r, "\psi"] & {|\mathbb{C}|}
\end{tikzcd}$$

$$\therefore \psi i' \in K_{id}(sk(\mathbb{C}))$$

$$\therefore (\psi i')(1) = \psi(i'(1)) = \psi(1);$$

$$\therefore \psi(1) = id_0^{(\mathbb{C})}$$

This is the requirement on ψ; there is a (unique) such ψ.

This means: $sk(\mathbb{C}) \models Ax_0$.

$\boxed{Ax_1}$ ::

($Ax_1 = Ax_{idun}$ = ax for the uniqueness for the identity arrow)

$$Ax_1 : \underbrace{\begin{tikzcd} 0 \arrow[r, shift left, "id_0"] \arrow[r, shift right, "id'_0"'] & 0 \end{tikzcd}}_{A_0} \longrightarrow \underbrace{\begin{tikzcd} 0 \arrow[r, "id_0"] & 0 \end{tikzcd}}_{A_1}$$

$Ob\,|A_0| = \{0\}$

$Arr\,|A_0| = \{1, 2\}$

$K_{id}(A_0) = \{\underbrace{[1 \mapsto 1], [1 \mapsto 2]}_{\text{both maps } \overline{K}_{id} \to |A_0|}\}$

$K_{id}(A_1) = \{[1 \mapsto 1]\}$

Again, Ax_1 is the unique morph with the given source and target

Note: $sk(\mathbb{C}) = Ax_1$; (...)

$\boxed{Ax_2}$ ::

($Ax_2 = Ax_{compex}$ = ax for the existence of composites)

$$Ax_2 : \begin{tikzcd} & 1 \arrow[dr, "4"] & \\ 0 \arrow[ur, "3"] & & 2 \end{tikzcd} \longrightarrow \underbrace{\begin{tikzcd} & 1 \arrow[dr, "4"] & \\ 0 \arrow[ur, "3"] \arrow[rr, "5"'] & \arrow[u, phantom, "="] & 2 \end{tikzcd}}_{A_1}$$

$K_{comm}(A_1) = \{id_{\overline{K}_{comm}}\}$

$\boxed{Ax_3}$ :: $\quad$ ($Ax_3 = Ax_{compun}$: ax for the uniqueness of composites)

Ax_3 :

$$\boxed{\begin{tikzcd} & 1 \arrow[dr, "4"] & \\ 0 \arrow[ur, "3"] \arrow[rr, bend left, "5"] \arrow[rr, bend right, "6"'] & & 2 \end{tikzcd} \quad \begin{matrix} \text{"}4 \circ 3 = 5\text{"} \\ \text{"}4 \circ 3 = 6\text{"} \end{matrix}} \longrightarrow \overbrace{\boxed{\begin{tikzcd} & 1 \arrow[dr, "4"] & \\ 0 \arrow[ur, "3"] \arrow[rr, bend right, "5"'] & \arrow[u, phantom, "\circledequal"] & 2 \end{tikzcd}}}^{A_1}$$

$K_{comm}(A_0) = \left\{ \begin{pmatrix} 345 \\ 345 \end{pmatrix}, \begin{pmatrix} 345 \\ 346 \end{pmatrix} \right\}$

$K_{comm}(A_1) = \{ id_{\overline{K}_{comm}} \}$

$\underline{Ax_4}$ :: $\quad$ ($A_{idcommleft}$)

$$\boxed{0 \xrightarrow{2} 1} \longrightarrow \underbrace{\boxed{\begin{tikzcd} & 0 \arrow[dr, "2"] & \\ 0 \arrow[ur, "id_0"] \arrow[rr, bend right, "2"'] & \arrow[u, phantom, "\circledequal"] & 1 \end{tikzcd}}}_{A_1}$$

$|A_1| = \boxed{3 \circlearrowright 0 \longrightarrow 1}$

$K_{id}(A_1) = \{ [1 \mapsto 3] \}$

$(\overline{K}_{id} \to |A_1|)$

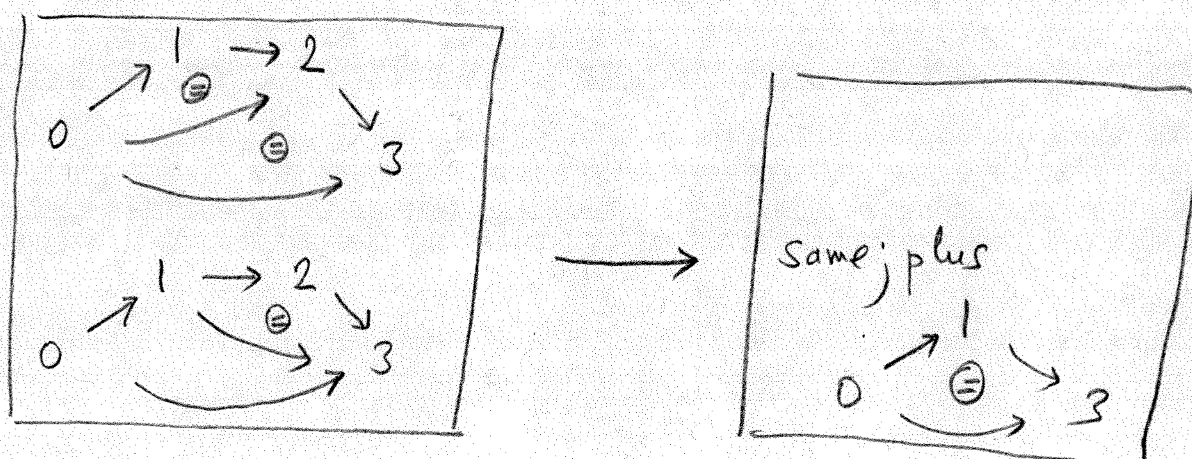
$K_{comm}(A_1) = \left\{ \begin{pmatrix} 012345 \\ 001322 \end{pmatrix} \right\}$ (get it?)

$(\overline{K}_{comm} \to |A_1|)$

$\boxed{Ax_5}$ ::

'dual' of Ax_4

$\boxed{Ax_6}$ :: $Ax_5 = A_{\text{associativity}}$



(If you see two '$0 \to 1$''s, they are the <u>same</u> arrow.)

Proposition For $sk : \mathrm{Cat} \longrightarrow \mathrm{SkCat}$,

$S \in \mathrm{SkCat}$

$S = sk(\mathbb{C})$, some $\mathbb{C}$

$\iff S \models \mathbb{R}_{cat} = \{Ax_0, \dots, Ax_6\}$

Two things to show:

1) $\mathbb{C} \in Cat \Rightarrow sk(\mathbb{C}) \models \mathbb{R}_{cat}$

'soundness' of the axiomatization

2) $S \in SkCat$, & $S \models \mathbb{R}_{cat}$

$\Rightarrow S = sk(\mathbb{C})$ for some $\mathbb{C} \in Cat$

'completeness' of the axiomatization

(Exercises).

[3] The deductive calculus

We return to the context of page [4]. We have a category $\mathbb{S}$, and a class $\mathbb{R}$ of arrows of $\mathbb{S}$. I define what it means for an arrow $\varphi : S \to T$ in $\mathbb{S}$ to be *deducible* from $\mathbb{R}$, in symbols

Definition 1: $\mathbb{R} \vdash \varphi$

The definition is an inductive one, with clauses 1), 2), 3) and 4):

1) For every 'axiom' $\alpha \in \mathbb{R}$, and for $\alpha :=: \emptyset \xrightarrow{?} \emptyset$

$$\mathbb{R} \vdash \alpha.$$

2) For every pushout diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "f"'] & T \arrow[d, "g"] \\ \hat{S} \arrow[r, "\hat{\varphi}"'] & \hat{T} \end{tikzcd}$$

$$\mathbb{R} \vdash \varphi \quad \Rightarrow \quad \mathbb{R} \vdash \hat{\varphi}.$$

3) Whenever $S \xrightarrow{\varphi} T \xrightarrow{\psi} U$,

$$\mathbb{R} \vdash \varphi \ \& \ \mathbb{R} \vdash \psi \quad \Rightarrow \quad \mathbb{R} \vdash \psi \circ \varphi.$$

4) Whenever $S \xrightarrow{\varphi} T \xrightarrow{\tau} U$,

$$\mathbb{R} \vdash \tau \circ \varphi \quad \Rightarrow \quad \mathbb{R} \vdash \varphi.$$

(It is understood that $\{\varphi : \mathbb{R} \vdash \varphi\}$ is the least class X of arrows closed under the closure 'conditions' corresponding to the four clauses; e.g., if φ & ψ are as in 3), then $\varphi \in X$ & $\psi \in X \Rightarrow \psi \circ \varphi \in X$)

Remarks. In 1), $\emptyset \xrightarrow{?} \emptyset$ is the unique endomorphism of the initial object $\emptyset$ of \$ (which is thereby assumed to exist).

The effect of the presence of $\emptyset \xrightarrow{?} \emptyset$ among the axioms is that all isomorphisms are deducible, since, for any isomorphism $\varphi: S \to T$,

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ S & \xrightarrow[\varphi]{\cong} & T \end{array}$$

is a pushout diagram. If we do not include $\emptyset \xrightarrow{?} \emptyset$ among the axioms, and denote the resulting notion $\mathbb{R} \overset{\circ}{\vdash} \varphi$, then, as it is easy to see, we have

$$\mathbb{R} \vdash \varphi \iff \mathbb{R} \overset{\circ}{\vdash} \varphi \ \underline{\text{or}}\ \varphi \text{ is an isomorphism}$$

(where <u>or</u> is <u>not</u> necessarily an exclusive 'or'!).

Another way of including isomorphism (instead of talking about $\emptyset$) is that, under 4)

we say that $\{\varphi : \mathbb{R} \vdash \varphi\}$ is closed under

composition of finitely many composable

arrows – with allowing "finitely many" to

mean the empty sequence of arrows from S to S,

for any S (all of whose members, of course,

are deducible!), whose composition is

$$S \xrightarrow{id_S} S$$

Once we have identities, we get all isomorphisms

by pushouts.

From now on, I assume has all finite colimits: pushouts, initial object, coproducts. Later, I will need essentially all small colimits — and, in fact, I will need that $\mathbb{S}$ is locally finitely presentable (lfp; explanation later)

$\boxed{\text{Definition 2}}$ Let $\varphi : S \longrightarrow T$. A <u>deduction</u> of φ from $\mathbb{R}$ is a system of entities as follows:

a natural number n;

objects S_j for $j = 0, \ldots, n-1, n$;

objects A_i^0, A_i^1 for $i = 0, \ldots, n-1$;

arrows φ_i, α_i, σ_i^0, σ_i^1 for $i = 0, \ldots, n-1$;

and an arrow τ :

<u>Such that</u>:

(i) $S_0 = S$

(ii) $\tau : T \longrightarrow S_n$

(iii) $\alpha_i : A_i^0 \longrightarrow A_i^1$ belongs to $\mathbb{R}$

(iv)

$$\begin{tikzcd}
A_i^0 \arrow[r, "\alpha_i"] \arrow[d, "\sigma_i^0"'] & A_i^1 \arrow[d, "\sigma_i^1"] \\
S_i \arrow[r, "\varphi_i"'] & S_{i+1}
\end{tikzcd}$$

is a pushout diagram

and:

(v) $\varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_1 \circ \varphi_0 = \tau \circ \varphi$.

$$\begin{tikzcd}
S = S_0 \arrow[r, "\varphi_0"] \arrow[drr, "\varphi"'] & S_1 \arrow[r] & \cdots \arrow[r] & S_{n-1} \arrow[r, "\varphi_{n-1}"] & S_n \\
& & T \arrow[urr, "\tau"'] \arrow[u, phantom, "="] & &
\end{tikzcd}$$

the given φ

end of definition of 'deduction'

Proposition Assume $\mathbb{S}$ has all pushouts. Then for any $\mathbb{R} \subseteq \mathrm{Arr}(\mathbb{S})$ and $\varphi \in \mathrm{Arr}(\mathbb{S})$,

$$\boxed{\mathbb{R} \vdash \varphi \iff \text{there is a deduction of } \varphi \text{ from } \mathbb{R}}$$

The right-to-left implication '$\Leftarrow$' is immediate. The other implication, '$\Rightarrow$', is easy too — but here is the proof:

For 1), page $\boxed{12}$: assume: $\alpha : A \to B \in \mathbb{R}$.

We let: $n = 1$

$$S_0 = A, \quad S_1 = B$$

$$A_0^0 = A, \quad A_0^1 = B$$

$$\varphi_0 = \alpha, \quad \alpha_0 = \alpha, \quad \sigma_0^0 = id_A, \quad \sigma_0^1 = id_B$$

$$\tau = id_B.$$

We have a deduction.

For 2): Assume the pushout diagram at 2), p $\boxed{12}$,

and assume φ has the deduction from $\mathbb{R}$ displayed in the def'n.

We construct a deduction of $\hat{\varphi}$ from $\mathbb{R}$ as follows.

Let: $\hat{S}_0 = \hat{S}$, and $(\hat{\sigma}_0 : S_0 \longrightarrow \hat{S}_0) \underset{\text{def}}{=} (f : S_0 \longrightarrow \hat{S})$

Inductively, we construct, for $i = 0, \dots, n-1$, the pushout diagrams:

$$\begin{tikzcd} S_i \arrow[r, "\varphi_i"] \arrow[d, "\sigma_i"'] & S_{i+1} \arrow[d, "\sigma_{i+1}"] \\ \hat{S}_i \arrow[r, "\hat{\varphi}_i"'] \arrow[ur, phantom, "\ulcorner \text{p.o.}"] & \hat{S}_{i+1} \end{tikzcd} \qquad (*)$$

(we construct the object $\hat{S}_{i+1}$, and the arrows $\hat{\varphi}_i$, σ_{i+1} from σ_i defined earlier, using the given data in $\varphi_i : S_i \longrightarrow S_{i+1}$)

With $\Phi \underset{\text{def}}{=} \varphi_{n-1} \circ \cdots \circ \varphi_0$, $\hat{\Phi} = \hat{\varphi}_{n-1} \circ \cdots \circ \hat{\varphi}_0$

we now have

$$\begin{tikzcd} S = S_0 \arrow[r, "\Phi"] \arrow[d, "f = \sigma_0"'] & S_n \arrow[d, "\sigma_n"] \\ \hat{S} = \hat{S}_0 \arrow[r, "\hat{\Phi}"'] \arrow[ur, phantom, "\ulcorner \text{p.o}"] & \hat{S}_n \end{tikzcd} \qquad (**)$$

a <u>pushout</u> diagram.

NB: we are using here the following simple – and basic – laws for pushouts in general:

Consider the commutative squares

$$\begin{tikzcd} \cdot \arrow[r, "g_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "1"] & \cdot \arrow[r, "g_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "2"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "h_0"'] & \cdot \arrow[r, "h_1"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "g_1 g_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "3"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "h_1 h_0"'] & \cdot \end{tikzcd}$$

We have 1 & 2 pushouts $\Rightarrow$ 3 pushout

1 & 3 pushouts $\Rightarrow$ 2 pushout

For the opposite category, these laws ~~be~~ become laws for pullbacks – and in that form, they can be verified for Set, the category of sets, and conclude that the laws hold in general $\lrcorner$

Next: we define the arrow $\hat{\tau} : \hat{T} \longrightarrow \hat{S}_n$

such that

$$\sigma_n \circ \tau = \hat{\tau} \circ g$$

and

$$\hat{\Phi} = \hat{\tau} \circ \hat{\varphi}$$

Consider the diagram:

$$\begin{tikzcd}
S = S_0 \arrow[rr, "\Phi"] \arrow[dr, "\varphi"] \arrow[dd, "f=\sigma_0"'] & & S_n \arrow[dd, "\sigma_n"] \\
& T \arrow[ur, "\tau"] \arrow[dd, "g"] & \\
\hat{S} = \hat{S}_0 \arrow[rr, dashed, "\hat\Phi"] \arrow[dr, "\hat\varphi"'] & & \hat{S}_n \\
& \hat{T} \arrow[ur, "\hat\tau"'] &
\end{tikzcd}$$

The diagram is commutative. The definition of $\hat\tau$ uses that $\hat{T}$ is a pushout, and that we have

$$\hat\Phi \circ \sigma_0 \overset{?}{=} \sigma_n \circ \tau \circ \varphi$$

$$(1) \, \| \qquad \sigma_n \circ \Phi \qquad \| \, (2)$$

(2) is because $\Phi = \tau\varphi$, by assumption on the deduction of φ; (1) is because of (**), p $\boxed{16}$. So, our diagram is in fact (completely) commutative.

Further, we take

$$
\begin{tikzcd}
A_i \arrow[r, "\alpha_i"] \arrow[d, "\sigma_i^0"'] \arrow[dd, bend right=60, "\hat{\sigma}_i^0 \underset{def}{=} \sigma_i \sigma_i^0"'] & A_{i+1} \arrow[d, "\sigma_i^1"] \arrow[dd, bend left=60, "\hat{\sigma}_i^1 \underset{def}{=} \sigma_{i+1} \sigma_i^1"] \\
S_i \arrow[r, "\varphi_i"] \arrow[d, "\sigma_i"'] \arrow[ur, phantom, "\ulcorner p.o."] & S_{i+1} \arrow[d, "\sigma_{i+1}"] \\
\hat{S}_i \arrow[r, "\hat{\varphi}_i"'] \arrow[ur, phantom, "\ulcorner p.o."] & \hat{S}_{i+1}
\end{tikzcd}
$$

and obtain the pushout

$$
\begin{tikzcd}
A_i \arrow[r, "\alpha_i"] \arrow[d, "\hat{\sigma}_i^0"'] & A_{i+1} \arrow[d, "\hat{\sigma}_i^1"] \\
\hat{S}_i \arrow[r] \arrow[ur, phantom, "\ulcorner p.o."] & \hat{S}_{i+1}
\end{tikzcd}
$$

Then: the items

$$
\left\{
\begin{array}{l}
n \\
\hat{S}_j \\
A_i^0, A_i^1 \\
\hat{\varphi}_i, \alpha_i, \hat{\sigma}_i, \hat{\sigma}_i^1 \\
\tau
\end{array}
\right.
\qquad
\left(
\begin{array}{l}
j \in [n+] \\
i \in [n-1]
\end{array}
\right)
$$

form a deduction of $\hat{S} \xrightarrow{\hat{\varphi}} \hat{T}$.

For 3): Let's have the arrows

$$S \xrightarrow{\varphi} T \xrightarrow{\psi} U$$

and suppose

$$\left\{\begin{array}{l} n \\ S_j \quad (j \in [n]) \\ \left.\begin{array}{l} A_i^0,\ A_i^1 \\ \varphi_i,\ \alpha_i,\ \sigma_i^0,\ \sigma_i^1 \end{array}\right\} (i \in [n-1]) \\ \tau \end{array}\right. \quad \text{resp.} \quad \left\{\begin{array}{l} m \\ T_k \quad (k \in [m]) \\ \left.\begin{array}{l} B_\ell^0,\ B_\ell^1 \\ \psi_\ell,\ \beta_\ell,\ \rho_\ell^0,\ \rho_\ell^1 \end{array}\right\} \ell \in [m-1] \\ \theta \end{array}\right.$$

constitutes a deduction for φ, resp. for ψ.

We have $T = T_0$; let $\tau_0 \underset{\text{def}}{=} \tau : T_0 \longrightarrow S_n$.

Inductively for $p = n, \ldots, n+m-1$, we define

S_{p+1}, $\varphi_p : S_p \to S_{p+1}$ and $\tau_{p-n+1} : T_{p-n+1} \longrightarrow S_{p+1}$ by

the pushout

$$\begin{tikzcd} S_p \arrow[r, "\varphi_p"] & S_{p+1} \\ T_{p-n} \arrow[u, "\tau_{p-n}"] \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & T_{p-n+1} \arrow[u, "\tau_{p-n+1}"'] \end{tikzcd}$$

Now, we have $\quad S_q \quad$ for $\quad q = 0, \dots, n, n+1, \dots, n+m$.

and $\quad S_p \xrightarrow{\varphi_p} S_{p+1} \quad$ for $\quad p = 0, \dots, n, \dots, n+m-1$.

Combine the previous pushouts with the assumed pushouts:

for ψ:

$$\boxed{\sigma_p^0 \underset{\mathrm{def}}{=} \tau_{p-n} \circ \rho_{p-n}^0}$$

$$\begin{tikzcd}
S_p \arrow[rrr, "\varphi_p"] & & & S_{p+1} \\
& T_{p-n} \arrow[ul, "\tau_{p-n}"'] \arrow[r, "\psi_{p-n}"] & T_{p-n+1} \arrow[ur, "\tau_{p-n+1}"] & \\
& B_{p-n}^0 \arrow[u, "\rho_{p-n}^0"'] \arrow[r, "\beta_{p-n}"'] \arrow[uul, bend left] \arrow[ul, phantom, "="] & B_{p-n}^1 \arrow[u, "\rho_{p-n}^1"] \arrow[uur, bend right] \arrow[ur, phantom, "="] &
\end{tikzcd}$$

$$\boxed{\sigma_p^1 \underset{\mathrm{def}}{=} \tau_{p-n+1} \circ \rho_{p-n}^1}$$

Let $\quad A_p^0 \underset{\mathrm{def}}{=} B_{p-n}^0, \quad A_p^1 \underset{\mathrm{def}}{=} B_{p-n}^1, \quad \alpha_p \underset{\mathrm{def}}{=} \beta_{p-n}$

for $p = n, \dots, n+m-1$.

Then $\quad n+m$

$$S_q \quad (q = 0, \dots, n, n+1, \dots, n+m)$$

$$A_p^0, \ A_p^1$$

$$\varphi_p, \ \alpha_p, \ \sigma_p^0, \ \sigma_p^1$$

and $\quad \sigma' \underset{\mathrm{def}}{=} \tau_m \circ \theta$

constitute a deduction of

$$S \xrightarrow{\psi \circ \varphi} U$$

Compare:

$$\begin{tikzcd}
S = S_0 \arrow[rr, "\Phi"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "="] & S_n \arrow[rr, "\Sigma"] & \arrow[d, phantom, "="] & S_{n+m} \\
& T \arrow[ur, "\tau"] \arrow[rr, "\Psi"] \arrow[dr, "\psi"'] & \arrow[d, phantom, "="] & T_m \arrow[ur, "\tau_m"'] & \\
& & U \arrow[ur, "\theta"'] \arrow[uurr, bend right, "\xi'"'] & &
\end{tikzcd}$$

where $\Phi = \varphi_{n-1} \circ \cdots \circ \varphi_0$, $\quad \Psi = \psi_{m-1} \circ \cdots \psi_0$

and $\quad \Sigma = \varphi_{n+m} \circ \cdots \circ \varphi_n$.

For 4), p $\boxed{12}$: This is trivial. If we have a deduction for $\tau \circ \varphi$, giving us

$$\begin{tikzcd}
S \arrow[rrr, "\Phi"] \arrow[dr, "\varphi"'] & & \arrow[d, phantom, "="] & U \\
& T \arrow[dr, "\tau"'] & & \\
& & V \arrow[uur, "\theta"'] &
\end{tikzcd}$$

then the same deduction with the last item θ replaced by $\theta \circ \tau$ will provide a deduction for φ.

What we've shown above is that the class

$$\{\varphi : \text{there is a deduction of } \varphi \text{ from } \mathbb{R}\}$$

is closed under the closure conditions 1) to 4).
— and that shows the implication '$\Rightarrow$' in the Proposition (p 14)

$\square$ Proposition (p 14)

$$\boxed{\begin{array}{l}\underline{\text{Soundness theorem}}\\[1em] \qquad \mathbb{R} \vdash \varphi \quad \Rightarrow \quad \mathbb{R} \models \varphi\end{array}}$$

Again, we show that

$$\{\varphi : \mathbb{R} \vdash \varphi\}$$

is closed under 1) to 4).

Re 1): This is obvious. Note that always

$$S \models \emptyset \xrightarrow{?} \emptyset.$$

$$(S \in Ob(\mathbb{S}))$$

<u>Re 2)</u>: Take the pushout as at 2), p 12, and assume $\mathbb{R} \models \varphi$. To show $\mathbb{R} \overset{}{\underset{?}{\models}} \hat{\varphi}$, let $\mathcal{U} \in \mathcal{S}$ such that $\mathcal{U} \models \mathbb{R}$; let $\hat{S} \xrightarrow{h} \mathcal{U}$; we want $k : \hat{T} \to \mathcal{U}$ such that:

$$
\begin{tikzcd}
\hat{S} \arrow[rr, "\hat{\varphi}"] \arrow[dr, "h"'] & {} \arrow[d, phantom, "="] & \hat{T} \arrow[dl, dashed, "?k"] \\
& \mathcal{U} &
\end{tikzcd}
$$

But since $\mathbb{R} \models \varphi$, $\mathcal{U} \models \mathbb{R}$, and $h \circ f : S \to \mathcal{U}$ (instantiating S in $\mathcal{U}$), we have $l : T \to \mathcal{U}$ such that

$$
\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, "f"'] & T \arrow[ddl, bend left, "l"] \arrow[dl, phantom, "="] \\
\hat{S} \arrow[d, "h"'] & \\
\mathcal{U} &
\end{tikzcd}
$$

Using the pushout that gives $\hat{T}$ (see p. 12), we get $k : \hat{T} \to \mathcal{U}$ such that $\boxed{k \circ \hat{\varphi} = h}$ (and $k \circ g = l$):

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, "f"'] & T \arrow[d, "g"] \arrow[ddl, bend left=60, "l"] \\
\hat{S} \arrow[r, "\hat{\varphi}"] \arrow[dr, "h"'] \arrow[r, phantom, shift right=3ex, "="] & \hat{T} \arrow[d, dashed, "k"'] \\
& U
\end{tikzcd}$$

which is what we wanted

<u>Re 3)</u>: This is just the following

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[dr, "\forall"'] \arrow[dr, phantom, shift left=3ex, "="] & T \arrow[r, "\psi"] \arrow[d, dashed, "\exists \; \forall"] \arrow[dr, phantom, shift left=3ex, "="] & U \arrow[dl, dashed, "\exists"] \\
& V &
\end{tikzcd}$$

Re 4):

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[dr, "\forall"'] \arrow[dr, phantom, shift left=3ex, "="] & T \arrow[r, "r"] \arrow[d, dashed, "l \overset{\text{def}}{=} k \circ r"] & U \arrow[dl, dashed, "\exists k"] \\
& V &
\end{tikzcd}$$

<u>Reminders on</u> some general concepts:

⊙ Recall: <u>directed</u> <u>colimits</u>, e.g. from the 'Coherent completeness' notes.

⊙ $\mathbb{S}$: a category (locally small).

An object X of $\mathbb{S}$ is $\boxed{\text{finitely presentable}}$ $\boxed{(\text{fp})}$, or, even, $\boxed{\text{finite}}$ (although this is less usual...) if the following is true. Given any directed diagram

$$A : I \longrightarrow \mathbb{S}$$

($A_i = A(i)$, $a_{ij} = A(i \le j)$ as usual)

and its colimit $A_\infty = \operatorname{colim} A$, with colimit cocone $\langle a_{i\infty} : A_i \to A_\infty \rangle_{i \in I}$,

we have the derived diagram

$$\hom_{\mathbb{S}}(X, A) : I \longrightarrow \mathrm{Set}$$

which is the composite

$$I \xrightarrow{A} \mathbb{S} \xrightarrow{\hom_{\mathbb{S}}(X,-)} \mathrm{Set}.$$

Take its colimit (in Set):

$$\operatorname{colim} \hom_{\mathcal{B}}(X, A)$$

also denoted as

$$\operatorname*{colim}_{i\in I} \hom_{\mathcal{B}}(X, A_i) \,. \qquad (*)$$

Note the following cocone on the diagram $\hom_{\mathcal{B}}(X, A)$:

$$\varphi_i : \hom_{\mathcal{B}}(X, A_i) \longrightarrow \hom_{\mathcal{B}}(X, A_\infty)$$

$$(X \xrightarrow[f]{} A_i) \longmapsto (X \xrightarrow[f]{} A_i \xrightarrow[a_{i\infty}]{} A_\infty)$$

The universal property of the colimit $(*)$ gives us

$$\varphi : \operatorname*{colim}_{i\in I} \hom_{\mathcal{B}}(X, A_i) \longrightarrow \hom_{\mathcal{B}}(X, \underbrace{\operatorname*{colim}_{i\in I} A_i}_{A_\infty})$$

uniquely determined by:

$$i \in I: \quad \begin{tikzcd} \operatorname*{colim}_{i\in I} \hom_{\mathcal{B}}(X, A_i) \arrow[rr, "\varphi"] & & \hom_{\mathcal{B}}(X, A_\infty) \\ & \hom_{\mathcal{B}}(X, A_i) \arrow[ul, "\gamma_i"] \arrow[ur, "\varphi_i"'] \arrow[u, phantom, "="] & \end{tikzcd}$$

where ~~the~~ γ_i are the colimit coprojections for the colimit $(*)$.

I say: X is $\boxed{\text{finite}}$ if the "canonical" map, the above φ, of sets:

$$\boxed{\operatorname{colim}_i \hom_{\mathbb{S}}(X, A_i) \xrightarrow{\cong} \hom_{\mathbb{S}}(X, \operatorname{colim}_i A_i)}$$

is an <u>isomorphism</u>.

① Category $\mathbb{S}$ is said to be $\boxed{\text{finitely accessible}}$ ~~(realeph zero accessible)~~ if

(i) the colimit of every small directed diagram in $\mathbb{S}$ exists;

(ii) for a <u>small</u> set $\mathbb{X}$ of finite objects, every object S of $\mathbb{S}$ is the colimit of a directed diagram $X : I \to \mathbb{S}$ for which $X_i \in \mathbb{X}$ for all $i \in I$.

⊙ Category $\mathbb{S}$ is said to be locally finitely presentable, lfp, or $\boxed{\text{locally finite}}$ if it is finitely accessible, and: (∗) it has all small colimits

<u>Remark</u>. Instead of (∗), it suffices to require that $\mathbb{S}_{fin}$, the full subcategory of $\mathbb{S}$ on the finite objects has all <u>finite</u> colimits. Also, instead of (∗), we may, equivalently, require that $\mathbb{S}$ has all small (projective) <u>limits</u>.

⊙ For brevity, an arrow $A \xrightarrow{\alpha} B$ is called <u>finite</u> if both A & B are finite objects. $U \xrightarrow{\varphi} V$ is called <u>relatively finite</u> if there is a pushout

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ U \arrow[r, "\varphi"'] & V \end{tikzcd}$$

with α a finite arrow.

General completeness theorem (GCT)

Let $\mathbb{S}$ be a locally finite category, $\mathbb{R}$ a set of finite arrows in $\mathbb{S}$, $\varphi: \xrightarrow{\varphi} V$ a relatively finite arrow.

Then

$$\mathbb{R} \models \varphi \quad \Rightarrow \quad \mathbb{R} \vdash \varphi$$

hence, because of soundness (p 23), we have

$$\mathbb{R} \models \varphi \quad \Leftrightarrow \quad \mathbb{R} \vdash \varphi$$

The proof of the GCT will be given in a later installment (it is given in MM / JPAA 1997, Part I, but this time I will give a different proof)

4 Sketch-specifying the coherent doctrine

We construct the sketch-category

$$\mathrm{Sk}\,\mathrm{Coh} = \mathrm{Sk}(\mathrm{Graph}, \mathrm{Ax}_0, \dots)$$

such that, for $\mathbb{R}_{\mathrm{Coh}} = \{\mathrm{Ax}_0, \dots\}$, we have a full-and-faithful, injective-on-objects embedding

$$\mathrm{sk} : \underbrace{\mathrm{Coh}}_{\substack{\text{the category of}\\ \text{small coherent categories}}} \longrightarrow \mathrm{Sk}\,\mathrm{Coh}$$

(which we treat as an inclusion), and Coh is specified by $\mathbb{R}_{\mathrm{Coh}}$ in Sk Coh, in the sense that for $S \in \mathrm{Sk}\,\mathrm{Coh}$,

$$S \in \mathrm{Coh} \iff S \models \mathbb{R}_{\mathrm{Coh}}.$$

As the notation already shows, our axioms continue the six ax's $Ax_0, \dots, Ax_6$ developed for specifying Cat in Sk Cat (see pages 5 to 10).

First, the "defined concept of isomorphism":

Let K_{iso} be a new specification name (in addition the two so far: K_{id} and K_{comm}).

Let

$$\overline{K}_{iso} :=: \boxed{0 \xrightarrow{2} 1}$$

and Ax_6, Ax_7 the following two sketch-entailments:

$\boxed{Ax_6}$ ::

$$Ax_6 : \underbrace{\boxed{0 \xrightarrow[2]{\cong} 1}}_{A_0} \longrightarrow \underbrace{\boxed{\begin{tikzcd} & 1 \arrow[dr, "3"] \arrow[d, phantom, "\circleddash"] & \\ 0 \arrow[ur, "2"] \arrow[rr, "id_0"'] & {} & 0 \\ 1 \arrow[rr, "id_1"] \arrow[dr, "3"'] & {} \arrow[d, phantom, "\circleddash"] & 1 \\ & 0 \arrow[ur, "2"'] & \end{tikzcd}}}_{A_1 = \underset{\text{target}}{t}(Ax_6)}$$

$$K_{iso}(A_0) = \{ id_{\overline{K}_{iso}} \}$$

Here : $$|A_1| = \boxed{\begin{tikzcd} 0 \arrow[loop left, "4"] \arrow[r, bend left, "2"] & 1 \arrow[l, bend left, "3"] \arrow[loop right, "5"] \end{tikzcd}}$$

$$\bar{K}_{id}(A_1) = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix} \right\}$$

$$\bar{K}_{comm}(A_1) = \left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 3 & 2 & 1 \end{pmatrix} \right\}$$

Ax_6 is the inclusion arrow.

$\boxed{Ax_7}$ ::

$$Ax_7 : \quad \boxed{\begin{matrix}\text{Same as} \\ t(Ax_6) \\ \text{previous}\end{matrix}} \longrightarrow \underbrace{\boxed{\begin{matrix}\text{Same} \\ \text{plus:} \\ \text{"2 is iso"}\end{matrix}}}_{A_1}$$

$$K_{iso}(A_1)$$

$$= \{ \text{inclusion } \bar{K}_{iso} \to |A_1| \}$$

<u>Note</u>: For any $(K_{id}, K_{comm}, K_{iso})$-sketch S,

$$S \models \{Ax_0, \ldots, Ax_6, Ax_7\} \quad \text{iff} \quad S \text{ is a}$$

category, and

$$\overline{K}_{iso} \xrightarrow{\varphi} |S| \quad \text{belongs to } K_{iso}(S)$$

iff

the arrow $\quad \varphi(2) : \varphi(0) \longrightarrow \varphi(1) \quad$ is an isomorphism in the category S.

Next, we specify finite limits.

Introduce $K_{(terminal)}$ and $K_{pullback}$, two new specification names, with

$\overline{K}_t = \boxed{0}$ (graph with a single object, no arrow)

and

$$\overline{K}_{pb} = \boxed{\begin{tikzcd} 0 \arrow[r, "4"] & 2 \\ 3 \arrow[u, "6"] \arrow[r, "7"'] & 1 \arrow[u, "5"'] \end{tikzcd}} \;;$$

two graphs.

Axioms follow:

[Ax_8 ::

$$Ax_8 : \underbrace{\boxed{\varnothing}}_{A_0} \longrightarrow \underbrace{\boxed{\underset{\text{terminal}}{0}}}_{A_1}$$

A_0 is the <u>empty graph</u>. $|A_1| = \bar{K}_t$,

and $K_t(A_1) = \{ id_{\bar{K}_t} \}$.

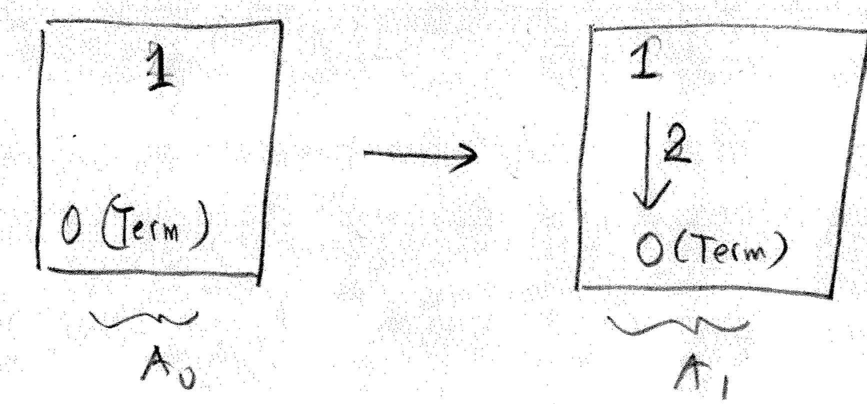
Ax_8 can be called $Ax_{termexi}$, the

axiom for the existence of a terminal object

$\boxed{Ax_9}$ :: $Ax_9 = Ax_{term\ univex} =$

axiom for the existence part of the
universal property of the terminal object

Ax_9 :



$|A_0| = \{ 0, 2 \}$ (no arrow)

$K_t(A_0) = \{$inclusion $\bar{K}_t \to |A_0| \}$

$0 \mapsto 0$

A_1 : self-explanatory.

$\boxed{Ax_{10}}$ :: $Ax_{10} = Ax_{\text{term univ un}}$ = axiom for the uniqueness part of the universal property of the terminal object

Ax_{10} :

$$\boxed{\begin{tikzcd} & 1 \arrow[dl, shift right, "2"'] \arrow[dl, shift left, "3"] \\ 0\ (\mathrm{Term}) & \end{tikzcd}} \longrightarrow \boxed{\begin{tikzcd} 1 \arrow[d, "2"] \\ 0\ (\mathrm{Term}) \end{tikzcd}}$$

(! the next one is unexpected!)

$\boxed{Ax_{11}}$ :: $Ax_{11} = Ax_{\text{term inv}}$ = axiom for the invariance-under-isomorphism of the concept "terminal object"

A_{11} :

$$\underbrace{\boxed{\begin{tikzcd} 1 \arrow[d, "2"'] \arrow[d, phantom, shift left=4, "\cong"] \\ 0\ (\mathrm{Term}) \end{tikzcd}}}_{A_0} \longrightarrow \underbrace{\boxed{\begin{tikzcd} 1\ \underline{\mathrm{Term}} \arrow[d, "2"'] \arrow[d, phantom, shift left=4, "\cong"] \\ 0\ (\mathrm{Term}) \end{tikzcd}}}_{A_1}$$

Here:

$$|A_0| = \boxed{\begin{tikzcd} 1 \arrow[d, "2"] \\ 0 \end{tikzcd}}$$

$$K_{iso}(A_0) = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} : \overline{K}_{iso} \longrightarrow |A_0| \right\}$$

see: p 32

$$K_{term}(A_0) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

For A_1: we also have $\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \overline{K}_{iso} \to |A_1|$ being an element of $K_{term}(A_1)$.

Of course, Ax_{11} expresses something that is true in any category — but why do we need it?

Proposition $\{Ax_0, \dots, Ax_{11}\}$ specify the concept of "category with a terminal object" among all $\{K_{id}, K_{comm}, K_{iso}, K_{term}\}$ - sketches.

I.e.; define, for a category $\mathbb{C}$,

$$sk_t(\mathbb{C})$$

to be the sketch S for which

$|S| = |\mathbb{C}|$, $K_{id}(S)$, $K_{comm}(S)$, $K_{iso}(S)$ as before, and

$$K_{term}(S) = \{ X \in Ob(\mathbb{C}) : X \text{ is a terminal object of } \mathbb{C} \}$$

$$\left(\text{rather: } \left\{ \ulcorner X \urcorner : \left\{ \begin{array}{l} \bar{K}_{term} \to |\mathbb{C}| \\ 0 \mapsto X \end{array} \right\} \;\middle|\; X \text{ is terminal} \right\} \right)$$

$\{0\}$

Then, for Cat_{term} the category of categories with (at least one) terminal object, and morphisms functors that map terminals to terminals, we have a full-faithful-injective on objects 'inclusion'

$$sk_t : Cat_t \longrightarrow Sk\,Cat_t$$

where $SkCat_t$ is the category of $\{K_{id}, \ldots, K_{term}\}$ sketches. The assertion is that, treating sk_t as an inclusion, we have that, for $S \in SkCat_t$,

$$S \in Cat_t \iff S \models \{Ax_0, \dots, Ax_{11}\}$$

– and Ax_{11} cannot be left out here.

<u>Proof of the proposition</u>:

The implication '$\Rightarrow$' is immediate.

To show '$\Leftarrow$', suppose $S \models \{Ax_0, \dots, Ax_{11}\}$

Then, first of all, S is a category ($S \upharpoonright SkCat = sk(\mathbb{C})$, $\mathbb{C} \in Cat$) – see Proposition, p. 10.

Moreover, by Ax_8, Ax_9 and Ax_{10} being true in S, S as a category has a terminal object, say X.

It is left to show that, for $Y \in Ob(S)$,

$\ulcorner Y \urcorner : \overline{K}_{term} \to |S| \; (0 \mapsto Y)$ belongs to the specification set $K_{term}(S)$

$$\iff$$

Y is a terminal object in the <u>category</u> S.

By Ax_9 and Ax_{10}, the implication '$\Rightarrow$'
is clear. Conversely, assume Y is a
terminal object in the category S.
Both X and Y are terminal :
therefore, there is an isomorphism

$$f : Y \xrightarrow{\cong} X$$

Let φ : source $(Ax_{11}) \longrightarrow S$
be sketch morphism $\varphi = \begin{pmatrix} 0 & 1 & 2 \\ X & Y & f \end{pmatrix}$. ($\varphi$
maps 1 to Y, 0 to X, and 2 to f.

φ is 'correct' as a sketch map, since
the specifications are respected: X is a terminal
object in the sketch S, and f, being an isomorphism
in the category S, f is an isomorphism in the
sketch S, by the force of $S \models Ax_7$ (see: p 33)

We have $S \models Ax_{11}$. Therefore, there is $\psi : \text{target}(A_{11}) \longrightarrow S$ such that

$\psi \circ Ax_{11} = \varphi$:

$$\begin{tikzcd}
\text{source}(Ax_{11}) \arrow[rr, "Ax_{11}"] \arrow[dr, "\varphi"'] \arrow[r, phantom, shift right=4ex, "="] & & \text{target}(Ax_{11}) \arrow[dl, dashed, "\psi"] \\
& S &
\end{tikzcd}$$

which shows that $\ulcorner \psi \urcorner$ indeed belongs to

$K_{term}(S)$.

$\square$ Proposition

(To be continued ...)