Strong conceptual completeness for Boolean coherent toposes

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What is strong conceptual completeness for first-order logic?

- A strong conceptual completeness statement for a logical doctrine is an assertion that a theory in this logical doctrine can be recovered from an appropriate structure formed by the models of the theory.
- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory $T$ from $\text{Mod}(T)$ equipped with structure induced by taking ultraproducts.
- Before we dive in, let’s look at a well-known theorem from model theory, with the same flavor, which Makkai’s result generalizes: the Beth definability theorem.
The Beth theorem

**Theorem.**

Let $L_0 \subseteq L_1$ be an inclusion of languages with no new sorts. Let $T_1$ be an $L_1$-theory. Let $F : \text{Mod}(T_1) \to \text{Mod}(\emptyset L_0)$ be the reduct functor. Suppose you know any of the following:

1. There is a $L_0$-theory $T_0$ and a factorization:
   \[
   \begin{array}{c}
   \text{Mod}(T_1) \\
   F \\
   \cong \\
   \text{Mod}(T_0)
   \end{array}
   \]

2. $F$ is full and faithful.
3. $F$ is injective on objects.
4. $F$ is full and faithful on automorphism groups.
5. $F$ is full and faithful on $\text{Hom}_{L_1}(M, M^U)$ for all $M \in \text{Mod}(T_1)$ and all ultrafilters $U$.
6. Every $L_0$-elementary map is an $L_1$-homomorphism of structures.

Then: (*) Every $L_1$-formula is $T_1$-provably equivalent to an $L_0$-formula.
Useful consequence of Beth’s theorem

Corollary.

Let $T$ be an $L$-theory, let $\bar{S}$ be a finite product of sorts. Let $X : \text{Mod}(T) \rightarrow \text{Set}$ be a subfunctor of $M \mapsto \bar{S}(M)$.

Then: if $X$ commutes with ultraproducts on the nose ("satisfies a Los’ theorem"), then $X$ was definable, i.e. $X$ is an evaluation functor for some definable set $\varphi \in \text{Def}(T)$.

Proof.

(Sketch): expand each model $M$ of $T$ by a new sort $X(M)$. Use commutation with ultraproducts to verify this is an elementary class. Then we are in the situation of $1 \implies (*)$ from Beth’s theorem.
How does strong conceptual completeness enter this picture?

- Plain old conceptual completeness (this was one of the key results of Makkai-Reyes) says that if an interpretation $I : T_1 \to T_2$ induces an equivalence of categories $\text{Mod}(T_1) \xrightarrow{I^*} \text{Mod}(T_2)$, then $I$ must have been a bi-interpretation. So, it proves $1 \implies (*)$, and therefore the corollary.
- Strong conceptual completeness is the following upgrade of the corollary.
Theorem.

Let $T$ be an $L$-theory. Let $X$ be any functor $\text{Mod}(T) \to \text{Set}$. Suppose that you have:

- for every ultraproduct $\prod_{i \to \mathcal{U}} M_i$ a way to identify $X(\prod_{i \to \mathcal{U}} M_i) \cong \prod_{i \to \mathcal{U}} X(M_i)$ ("there exists a transition isomorphism"), such that

- $(X, \Phi)$ preserves ultraproducts of models/elementary embeddings ("is a pre-ultrafunctor"), and also

- preserves all canonical maps between ultraproducts ("preserves ultramorphisms").

Then: there exists a $\varphi(x) \in T^\text{eq}$ such that $X \simeq \text{ev}_{\varphi(x)}$ as functors $\text{Mod}(T) \to \text{Set}$. (We call such $X$ an ultrafunctor.)
That is, the specified transition isomorphisms $\Phi_{(M_i)} : X(\prod_{i \to \mathcal{U}} M_i) \to \prod_{i \to \mathcal{U}} X(M_i)$ make all diagrams of the form

\[
\begin{array}{c}
X(\prod_{i \to \mathcal{U}} M_i) \xrightarrow{\Phi_{(M_i)}} \prod_{i \to \mathcal{U}} X(M_i) \\
\downarrow X(\prod_{i \to \mathcal{U}} f_i) \quad \quad \quad \quad \quad \quad \downarrow \prod_{i \to \mathcal{U}} X(f_i) \\
X(\prod_{i \to \mathcal{U}} N_i) \xrightarrow{\Phi_{(N_i)}} \prod_{i \to \mathcal{U}} X(N_i)
\end{array}
\]

commute ("transition isomorphism/pre-ultrafunctor condition").
What are ultramorphisms?

An ultragraph $\Gamma$ comprises:

- A directed graph whose vertices are partitioned into free nodes $\Gamma^f$ and bound nodes $\Gamma^b$.
- For any bound node $\beta \in \Gamma^b$, we assign a triple $\langle I, U, g \rangle \overset{\text{df}}{=} \langle I_\beta, U_\beta, g_\beta \rangle$ where $U$ is an ultrafilter on $I$ and $g$ is a function $g : I \rightarrow \Gamma^f$.
- An ultradiagram for $\Gamma$ is a diagram of shape $\Gamma$ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions $g$.
- A morphism of ultradiagrams (for fixed $\Gamma$) is just a natural transformation of functors which respects the extra data: the component of the transformation at a bound node is the ultraproduct of the components for the indexing free nodes.
Okay, but what are ultramorphisms?

**Definition.**

Let \( \text{Hom}(\Gamma, \mathcal{S}) \) be the category of all ultradiagrams of type \( \Gamma \) inside \( \mathcal{S} \) with morphisms the ultradiagram morphisms defined above. Any two nodes \( k, \ell \in \Gamma \) define evaluation functors \( (k), (\ell) : \text{Hom}(\Gamma, \mathcal{S}) \rightarrow \mathcal{S} \), by

\[
(k) \left( A \xrightarrow{\Phi} B \right) = A(k) \xrightarrow{\Phi_k} B(k)
\]

(resp. \( \ell \)).

An ultramorphism of type \( \langle \Gamma, k, \ell \rangle \) in \( \mathcal{S} \) is a natural transformation \( \delta : (k) \rightarrow (\ell) \).

It’s sufficient to consider the ultramorphisms which come from universal properties of colimits of products in \( \textbf{Set} \).
Strong conceptual completeness, II

Now, what’s changed between this statement and that of the useful corollary to Beth’s theorem?

- We dropped the *subfunctor* assumption! We don’t have such a nice way of knowing exactly how $X(M)$ is obtained from $M$. We only have the invariance under ultra-stuff. We’ve left the placental warmth of the ambient models and we’re considering some kind of abstract permutation representation of $\text{Mod}(T)$.

- Yet, if $X$ respects enough of the structure induced by the ultra-stuff, then $X$ must have been constructible from our models in some first-order way (“is definable”).

- (With this new language, the corollary becomes: ”strict sub-pre-ultrafunctors of definable functors are definable.”)
Strong conceptual completeness, III

Actually, Makkai proved something more, by doing the following:

- Introduce the notions of ultracategory and ultrafunctors by requiring all this extra ultra-stuff to be preserved.
- Develop a general duality theory between pretoposes (\( \text{Def}(T) \)) and ultracategories (\( \text{Mod}(T) \)) via a contravariant 2-adjunction ("generalized Stone duality").
- In particular, from this adjunction we get
  \[
  \text{Pretop}(T_1, T_2) \simeq \text{Ult}(\text{Mod}(T_2), \text{Mod}(T_1)).
  \]

Therefore, SCC tells us how to recognize a reduct functor in the wild between two categories of models—i.e., if there is some uniformity underlying a functor \( \text{Mod}(T_2) \to \text{Mod}(T_1) \) due to a purely syntactic assignment \( T_1 \to T_2 \). Just check if the ultra-structure is preserved!
Caveat. Of course, one has an infinite list of conditions to verify here.

- So the only way to actually do this is to recognize some kind of uniformity in the putative reduct functor which lets you take care of all the ultramorphisms at once.
- But it gives you another way to think about uniformities you need.
- It also gives you a way to check that something can never arise from any interpretation!
Important examples of ultramorphisms

Examples.

- **The diagonal embedding** into an ultrapower.
- **Generalized diagonal embeddings.** More generally, let $f : I \to J$ be a function, let $\mathcal{U}$ be an ultrafilter on $I$ and let $\mathcal{V}$ be the pushforward ultrafilter on $J$. Then for any $I$-indexed sequence of structures $(M_i)_{i \in I}$, there is a canonical map $\delta_f : \prod_{j \to \mathcal{V}} M_{f(i)} \to \prod_{i \to \mathcal{U}} M_i$ given by taking the diagonal embedding along each fiber of $f$. 


**Δ-functors induce continuous maps on automorphism groups**

- Why should we expect ultramorphisms to help us identify evaluation functors in the wild?
- Here’s an result which might indicate that knowing that they’re preserved tells us something nontrivial.

**Definition.**

Say that $X : \text{Mod}(T) \to \text{Mod}(T')$ is a Δ-functor if it preserves ultraproducts and diagonal maps into ultrapowers. Equip automorphism groups with the topology of pointwise convergence.

**Theorem.**

If $X$ is a Δ-functor from $\text{Mod}(T)$ to $\text{Mod}(T')$, then $X$ restricts to a continuous map $\text{Aut}(M) \to \text{Aut}(X(M))$ for every $M \in \text{Mod}(T)$. 
Proof.

- The topology of pointwise convergence is sequential, so to check continuity it suffices to check convergent sequences of automorphisms are preserved.

- If \( f_i \to f \) in Aut\((M)\), then since the cofinite filter is contained in any ultrafilter, \( \prod_{i \to \mathcal{U}} f_i \) agrees with \( \prod_{i \to \mathcal{U}} f \) over the diagonal copy of \( M \) in \( M^\mathcal{U} \). That is, \((\prod_{i \to \mathcal{U}} f_i) \circ \Delta_M = (\prod_{i \to \mathcal{U}} f) \circ \Delta_M\).

- Applying \( X \) and using that \( X \) is a \( \Delta \)-functor, conclude that \( \prod_{i \to \mathcal{U}} X(f_i) \) agrees with \( \prod_{i \to \mathcal{U}} X(f) \) over the diagonal copy of \( X(M) \) inside \( X(M)^\mathcal{U} \).

- For any point \( a \in X(M) \), the above says the sequence \( (X(f_i)(a))_{i \in I} =_{\mathcal{U}} (X(f)(a))_{i \in I} \).

- Since \( \mathcal{U} \) was arbitrary and the cofinite filter on \( I \) is the intersection of all non-principal ultrafilters on \( I \), we conclude that the above equation holds cofinitely. Hence, \( X(f_i) \to X(f) \).
\(\aleph_0\)-categorical theories

- A first-order theory \(T\) is \(\aleph_0\)-categorical if it has one countable model up to isomorphism.

- \(\aleph_0\)-categorical theories have only finitely many types in each sort. (Caveat: when I say “type”, I mean an atom in \(\mathcal{E}(T)\).)

- A theorem of Coquand, Ahlbrandt and Ziegler says that, given two \(\aleph_0\)-categorical theories \(T\) and \(T'\) with countable models \(M\) and \(M'\), a topological isomorphism \(\text{Aut}(M) \simeq \text{Aut}(M')\) induces a bi-interpretation \(M \simeq M'\).

- Since we know \(\Delta\)-functors induce continuous maps on automorphism groups, they’re a good candidate for definable functors.

- Boolean coherent toposes split into a finite coproduct of \(\mathcal{E}(T_i)\), where each \(T_i\) is \(\aleph_0\)-categorical.
A definability criterion for $\aleph_0$-categorical theories

Theorem.

Let $X : \text{Mod}(T) \to \text{Set}$. If $T$ is $\aleph_0$-categorical, the following are equivalent:

1. For some transition isomorphism, $(X, \Phi)$ is a $\Delta$-functor (preserves ultraproducts and diagonal maps).
2. For some transition isomorphism, $(X, \Phi)$ is definable.
A definability criterion for $\aleph_0$-categorical theories

Proof.

(Sketch.)

- One direction is immediate by SCC: definable functors are ultrafunctors are at least $\Delta$-functors.

- Let $M$ be the countable model. Use the lemma about $\Delta$-functors $(X, \Phi)$ inducing continuous maps on the automorphism groups (equivalently, $(X, \Phi)$ has the finite support property) to cover each $\text{Aut}(M)$-orbit of $X(M)$ by a projection from an $\text{Aut}(M)$-orbit of $M$. By $\omega$-categoricity, the kernel relation of this projection is definable, so we know that $X(M)$ looks like an (a priori, possibly infinite) disjoint union of types.

- By $\text{Aut}(M)^U$ orbit-counting, there are actually only finitely many types.

- Invoke the Keisler-Shelah theorem to transfer to all $N \models T$. 
A definability criterion for $\aleph_0$-categorical theories

**Corollary.**

Let $T$ and $T'$ be $\aleph_0$-categorical. Let $X$ be an equivalence of categories

$$\text{Mod}(T) \xrightarrow{X} \text{Mod}(T').$$

Then $X$ was induced by a bi-interpretation $T_1 \simeq T_2$ if and only if $X$ was a $\Delta$-functor.

In particular, Bodirsky, Evans, Kompatscher and Pinkser gave an example of two $\aleph_0$-categorical theories $T$, $T'$ with abstractly isomorphic but not topologically isomorphic automorphism groups of the countable model. This abstract isomorphism induces an equivalence $\text{Mod}(T) \simeq \text{Mod}(T')$ and since it can’t come from an interpretation, from the corollary we conclude that it fails to preserve an ultraproduc or a diagonal map was not preserved.
Exotic pre-ultrafunctors

In light of the previous result, a natural question to ask is:

**Question.**

*Is being a $\Delta$-functor enough for SCC? That is, do non-definable $\Delta$-functors exist?*

**Theorem.**

*The previous definability criterion fails for general $T$. That is:*

- There exists a theory $T$ and a $\Delta$-functor $(X, \Phi) : \text{Mod}(T) \to \text{Set}$ which is not definable.
- There exists a theory $T$ and a pre-ultrafunctor $(X, \Phi)$ which is not a $\Delta$-functor (hence, is also not definable.)
Exotic pre-ultrafunctors

Proof.

(Sketch.)

- Complete types won’t work, so take a complete type and cut it in half into two partial types, one of which refines the other. Define $X(M)$ to be the realizations in $M$ of the coarser one.
- Taking ultraproducts creates external realizations ("infinite/infinitesimal points") of either one.
- You can either try to construct a transition isomorphism which turns it into a pre-ultrafunctor (creating a non-$\Delta$ pre-ultrafunctor) or obtain one non-constructively (creating a non-definable $\Delta$-functor).
Future work

- Is the above $X(M)$ isomorphic to $\text{ev}_A$ for some $A \in \mathcal{E}(T)$?
- Which parts of Makkai’s ultra-data ensure $X : \text{Mod}(T) \to \text{Set}$ is $\text{ev}_A$ for $A \in \mathcal{E}$ and which parts make sure that $A$ is compact?
- How do ultramorphisms relate to the Awodey-Forssell duality?
- Conjecture: the pre-ultrafunctor part of the data ensures compactness after you get inside the classifying topos, i.e. if you start with $A \in \mathcal{E}$ and $\text{ev}_A$ is an ultrafunctor, then $A$ was compact.
- **Update:** this last conjecture is actually true!
Latest results:

**Theorem.**

Let $\mathcal{E}(T)$ be the classifying topos of a first-order theory. Let $B$ be an object of $\mathcal{E}(T)$. The following are equivalent:

1. $B$ is coherent.
2. $\text{ev}_B : \text{Mod}(T) \to \text{Set}$ is a pre-ultrafunctor.
3. The reduct functor $\text{Mod}(T[B]) \xrightarrow{I^*} \text{Mod}(T)$ is an equivalence, where $T[B]$ is $T$ with an additional sort for $B$ and all the induced definable structure on $B$ (“the graph of $\mathcal{E}(T)(y(-), B)$”) adjoined.
4. $\text{Mod}(\mathcal{E}(T)/B)$ is an ultracategory such that the forgetful functor $F : \text{Mod}(\mathcal{E}(T)/B) \to \text{Mod}(T)$ is an ultrafunctor and the functor $(\langle M, b \rangle \mapsto \{b\}) : \text{Mod}(\mathcal{E}(T)/B) \to \text{Set}$ is a strict ultrafunctor.
Thank you!