Limit closure of categories of domains

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http://www.math.mcgill.ca/barr/papers

Abstract

This continues (under a better title) the talk I gave three weeks ago. I give some of the proofs of claims made then.

Basic assumptions (recall)

- $\mathcal{A}$ is a category of domains.
- Every domain can embedded into a field that belongs to $\mathcal{A}$.
- $\mathcal{K}$ is the limit closure of $\mathcal{A}$ in commutative rings.
- $\mathcal{B}$ consists of the domains in $\mathcal{K}$.
- $\mathcal{SPR}$ is the category of semiprime rings.
- $\mathbf{Q}(D)$ is the field of fractions of the domain D.
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Construction of K and G (recall)

$K : \mathcal{SPR} \longrightarrow \mathcal{K}$ is the adjoint to the inclusion of $\mathcal{K}$ into the category of semiprime rings, easily shown to exist.
$G(D)$ is the meet of all objects of $\mathcal{B}$ that contain D.

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Why does G exist?

(Amalgamation) If D is a subdomain of both D_1 and D_2, then there is a field in $F \in \mathcal{A}$ that contains both D_1 and D_2.

$$\begin{tikzcd}
 & D_1 \arrow[rr, hook] & & \mathbf{Q}(D_1) \arrow[dr, hook] & & & \\
D \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, hook] & & \mathbf{Q}(D) \arrow[ur, hook] \arrow[dr, hook] & & T \arrow[r] & T/M \arrow[r, hook] & F \\
 & D_2 \arrow[rr, hook] & & \mathbf{Q}(D_2) \arrow[ur, hook] & & &
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where $T = \mathbf{Q}(D_1) \otimes_{\mathbf{Q}(D)} \mathbf{Q}(D_2)$ and $M \subseteq T$ is a maximal ideal and $F \in \mathcal{A}$ is a field.

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Computing $G(D)$

As a result, we can compute $G(D)$ by embedding D in a field $F \in \mathcal{A}$ and letting $G(D)$ be the meet of all $\mathcal{B}$-subobjects of F that contain D.

The inclusion $D \longrightarrow G(D)$ is epic in $\mathcal{SPR}$.

For suppose that $G(D) \underset{g}{\overset{f}{\rightrightarrows}} R$ are two maps that agree on D. Can suppose R is a field and even a field in $\mathcal{A}$, whence the equalizer of f and g will be a smaller $\mathcal{B}$-subobject that contains D

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Epimorphisms of fields

If $F \longrightarrow E$ is an epimorphism between fields, then E is a purely inseparable extension of F.

Proof: Factor the map as $F \longrightarrow F_1 \longrightarrow F_2 \longrightarrow E$ into a pure transcendental extension followed by a purely inseparable extension followed by a separable extension.

$F_2 \longrightarrow E$ is epic but as soon as it is proper there are non-trivial maps of E into its algebraic closure that fix F_2. Hence $F_2 = E$.

Since the embedding of F_2 into its perfect closure is epic, we can suppose F_2 is perfect.

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Epimorphisms of fields, cont'd

If F_1 were a proper extension of F, then it has many automorphisms that fix F. Let $\sigma \neq 1$ be one. For any $a \in F_2$, there is an integer k s.t. $a^{p^k} \in F_1$.
Then there is a unique element, call it $\overline{\sigma}(a) \in F_2$ s.t. $(\overline{\sigma}(a))^{p^k} = \sigma(a^{p^k})$. Then $\overline{\sigma}$ is an automorphism of F_2 that extends σ and therefore fixes F, a contradiction so that $F_1 = F$.
For any domain D, $G(D)$ is a subdomain of the perfect closure of $\mathbf{Q}(D)$. In characteristic 0, $G(D)$ is a subdomain of $\mathbf{Q}(D)$.

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Epimorphisms in $\mathcal{SPR}$

Suppose $f : R \hookrightarrow S$ is epic in $\mathcal{SPR}$. Then $\mathrm{Spec}(f)$ is injective.

Proof. Assume Q, Q' are primes of S s.t. $P = Q \cap R = Q' \cap R$.
Use the amalgamation property to form the diagram

$$\begin{tikzcd} R/P \arrow[r, hook] \arrow[d, hook] & S/Q \arrow[d, hook] \\ S/Q' \arrow[r, hook] & F \end{tikzcd}$$

with F a field. The square commutes so the two maps $S \longrightarrow S/Q \longrightarrow F$ and $S \longrightarrow S/Q' \longrightarrow F$ agree on R, but have different kernels.

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Properties of K

Canonical $R \longrightarrow K(R)$ is injective and epic in $\mathcal{SPR}$.

Proof: Embed R into a product $\prod F_i$ of fields, which can be assumed to lie in $\mathcal{A}$. The embedding $R \hookrightarrow \prod F_i$ factors through $K(R)$, by adjointness, and a first factor of an injection is an injection.

If $f, g : K(R) \longrightarrow S$ are two maps into a semiprime ring that agree on R, we can easily reduce to the case that S is a field in $\mathcal{A}$ and then it follows by adjointness.

$R \longrightarrow K(R)$ induces an injection $\mathrm{Spec}(K(R)) \longrightarrow \mathrm{Spec}(R)$.

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Properties of K, cont'd

Canonical $R \longrightarrow K(R)$ induces bijection $\mathrm{Spec}(K(R)) \longrightarrow \mathrm{Spec}(R)$.

We just saw it was injective. For surjectivity, suppose $P \subseteq R$ is prime. Embed $R/P \hookrightarrow F$, a field in $\mathcal{A}$. From adjointness, we have a square

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & K(R) \arrow[d] \\ R/P \arrow[r, hook] & F \end{tikzcd}$$

from which we see that the kernel of $K(R) \longrightarrow F$ is a prime lying over P.

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Order structure

Note, however, that this bijection is not an order isomorphism in general. Example: $\mathbf{Z} \longrightarrow K_{\mathrm{fld}}(\mathbf{Z})$.

If $P \subseteq R$, let $P^{@} \subseteq K(R)$ be the kernel of $K(R) \longrightarrow K(R/P)$. Since $K(R/P)$ is not generally a domain, there is no reason for $P^{@}$ to be prime.

We do have: If $P \subseteq Q$, then $P^{@} \subseteq Q^{@}$.

Use the diagonal fill-in in

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In the red case, we begin to see that the Spec bijection is an order isomorphism. The induced map preserves order and so is the inverse once we see that $P^{@}$ is the inverse of P. That is that $P^{@} \cap R = P$. But this is immediate from

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David Dobbs's "Folklore" theorem

For rings $R \subseteq T$, we have that T is an integral extension of R if and only if whenever $R \subseteq S_1 \subseteq S_2 \subseteq T$,

1. $\mathrm{Spec}(S_2) \longrightarrow \mathrm{Spec}(S_1)$ is surjective; and
2. If $P \subseteq Q$ are primes of S_2 with $P \cap S_1 = Q \cap S_1$, then $P = Q$.

This obviously implies:

Assume $R \subseteq T$. If $R \subseteq S_1 \subseteq S_2 \subseteq T$ implies $\mathrm{Spec}(S_2) \longrightarrow \mathrm{Spec}(S_1)$ is bijective, then T is an integral extension of R.

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Integral extensions à la Zariski & Samuel

Theorem V.3 of Z&S and its corollary state the following:

Suppose S is an integral extension of R. Then any prime of R has a prime of S lying over it.

Moreover, if $P_1 \subseteq P_2$ are primes of R and Q_1 is a prime of S lying above P_1, then there is at least one prime Q_2 of S lying above P_2 and s.t. $Q_1 \subseteq Q_2$.

This obviously implies:

If $R \hookrightarrow S$ is both epic and integral, then $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is an order isomorphism.

This, together with Dobbs's theorem is the key to the connection between rougeosity and integrality.

At this point, most of what is needed to prove the 14 (now grown to 15) part theorem.

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DL-closed rings

We will be sketching the proof that $\mathcal{K}_{\mathrm{dom}}$ is the category of DL-closed rings. One thing that makes this case much easier is that when R is DL-closed and $P \subseteq R$ is a prime ideal, then R/P is DL-closed.

The proof that $\mathcal{K}_{\mathrm{ic}}$ is the category of (2,3)-closed rings is much harder and I will not attempt to do it in this talk.

The sheaf

The following construction is for the red case. There is a more more complicated one in the black case with less satisfactory results.

So suppose $\mathcal{A}$ is a red category of domains, $\mathcal{B}$ and $\mathcal{K}$ as before. Given a ring R (always assumed semiprime), we build a sheaf as follows. The base is $\mathrm{Spec}(R)$ with the domain topology and the stalk above the prime P is $G(R/P) = K(R/P)$.

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The topology

The problem is that the map $R \longrightarrow G(R/P)$ is not surjective so that we cannot simply topologize the union of the stalks by the finest topology for which each constant section is continuous.

Suffice it to say that there is a topology on the union of the stalks for which the constant functions are continuous. It has two important properties: products and sums of continuous sections are continuous and zero sets are open. This means that if σ is a section, then $\{P \mid \sigma(P) = 0\}$ is open. You might more readily expect it to be closed, but since $\mathrm{Spec}(R)$ is not T1, it isn't. The real reason it's open comes down to the fact that, given $r \in R$, $\{P \mid r \in P\}$ is open. This is just the zero set of the constant section at r.

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Global sections and K

$\Gamma(E) \in \mathcal{K}$. That is, the ring of global sections is in the limit closure.

I will not prove it, but it comes down to showing that the global sections of a sheaf can be constructed as a complicated limit starting with the stalks. Since the stalks belong to $\mathcal{K}$ so does the ring of global sections.

$\Gamma(E) \cong K(R)$ in such a way that the map $\zeta : R \longrightarrow \Gamma(E)$, defined by $\zeta(r)(P) = r + P$, is the adjunction morphism.

This holds in the red case and, under certain additional conditions, in others. It is not so easy, however.

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The main theorem

Recall that a ring is DL-closed if whenever $r^3 = s^2$ and r is a square mod every prime, then there is a unique t s.t. $t^2 = r$ and $t^3 = s$.

Let R be a commutative semiprime ring. Then the following are equivalent:

1. R is DL-closed
2. R is isomorphic, under the canonical map, to the ring of global sections of the sheaf E.
3. R is isomorphic to a ring of global sections of a sheaf whose stalks are domains.
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Local representation of sections

Before beginning the proof sketch, we must look more closely at the sheaf. If $\gamma \in \Gamma(E)$ and U is a compact open subset of $\mathrm{Spec}(R)$, we will say that the element $r \in R$ **represents** γ on U if for all $P \in U$, $\gamma(P) = r + P$, that is the image of r in R/P.

We say that $\gamma = r$ if r on U represents γ on U and that $\gamma = r$ if r represents γ on all of $\mathrm{Spec}(R)$.

We say that $(r_1, \ldots, r_n; U_1, \ldots, U_n)$ represents γ on $U = \bigcup U_i$ if r_i represents γ on U_i. Among other things, this requires that the U_i be compact and open.

Suppose that γ is a section and U is a compact open subset of $\mathrm{Spec}(R)$ such that $\gamma(P) \in R$ for all $P \in U$, then γ is locally representable on U.

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Local representations, cont'd

If $(U_1, U_2, \ldots, U_n; r_1, r_2, \ldots, r_n)$ represents γ on $U = U_1 \cup U_2 \cup \cdots \cup U_n$, then for all sufficiently large w there is an $a_w \in R$ such that $(U_1 \cup U_2, \ldots, U_n; a_w, (r_3 - r_1)^w, \ldots, (r_n - r_1)^w)$ represents $(\gamma - r_1)^w$ on U.

The relevance is this. We are going to be showing that when R is DL-closed, every global section is representable. Well, γ is representable iff $\gamma - r_1$ is. Second, just consider the case $n = 2$. Then this says that all sufficiently large powers of $\gamma - r_1$ are representable. If R is DL-closed, θ representable mod every prime and if θ^2 and θ^3 are representable, then θ is. But if θ^w is representable for all sufficiently large w, then so are $\theta^{2(w-1)}$ and $\theta^{3(w-1)}$ and hence θ^{w-1} and eventually θ.

An important observation is that since open sets are up-closed and the set of P for which $\gamma(P) = r$ is open, it follows that $P \subseteq Q$ implies $\gamma(Q) = r$.

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Proof: Assume that $r_1 = 0$. For $i = 1, 2$, let $J_i = \bigcap\{P \mid P \in U_i\}$. We claim that $r_2^w \in J_1 + J_2$ for all sufficiently large w. This is equivalent to showing that the image of r_2 belongs to every prime of $R/(J_1 + J_2)$ or equivalently, that r_2 belongs to every prime of R that contains both J_1 and J_2.

So suppose that Q is such a prime. We can show (handout) that there exist $P_1 \in U_1$ and $P_2 \in U_2$ with $P_1 \subseteq Q$ and $P_2 \subseteq Q$. But $\gamma(P_1) = r_1 = 0$, which implies that $\gamma(Q) = 0$ since sections will agree on an open set and every open set is up-closed in the domain topology. Similarly $\gamma(P_2) = r_2$ which implies that $\gamma(Q) = r_2 = 0$ and thus $r_2 \in Q$, as claimed. For sufficiently large w, we can write $r_2^w = a_w + b_w$ with $a_w \in J_1$ and $b_w \in J_2$.

If $P \in U_1$, we have that $0 = \gamma^w(P) = a_w$ since $a_w \in J_1 \subseteq P$. If $P \in U_2$, then $\gamma^w(P) = a_w + b_w = a_w$ since $b_w \in J_2 \subseteq P$ and so we see that $\gamma^w = a_w$ on all of $U_1 \cup U_2$ as required.

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Proof of main theorem

- DL-1 $\Rightarrow$ DL-2: This is the essence of the preceding development
- DL-2 $\Rightarrow$ DL-1: Assume that $a, b \in R$ are such that a has a square root mod every prime ideal and that $a^3 = b^2$. Then mod every prime ideal P, there exists a unique c_P such that mod P, we have $c_P^2 = a$ and $c_P^3 = b$. Since $Z(c_P^2 - a) \cap Z(c_P^3 - b)$ is open in the domain topology, these equations hold in a neighbourhood of P and the elements c must agree on overlaps by uniqueness. So they determine a section γ. But by DL-2, $\gamma \in R$ and so R satisfies DL-1.
- DL-2 $\Rightarrow$ DL-3 $\Rightarrow$ DL-4: The first is obvious, while the second is a consequence of the fact that the ring of global sections is in the limit closure of the stalks.
- DL-4 $\Rightarrow$ DL-2: We are in the red case.

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