TRACKING THE SNAKE TO ITS LAIR

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The snake lemma states that given any commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1' \arrow[r, "f_1"] \arrow[d, "u"] & A_1 \arrow[r, "g_1"] \arrow[d, "v"] & A_1'' \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & A_0' \arrow[r, "f_0"'] & A_0 \arrow[r, "g_0"'] & A_0'' \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

in an abelian category in which both rows are exact, there is canonical map $\ker w \longrightarrow \operatorname{coker} u$ so that the sequence

$$0 \longrightarrow \ker u \longrightarrow \ker v \longrightarrow \ker w \longrightarrow \operatorname{coker} u \longrightarrow \operatorname{coker} v \longrightarrow \operatorname{coker} w \longrightarrow 0$$

is exact.

But it is also true that when $f : A' \longrightarrow A$ and $g : A \longrightarrow A''$ are arrows in an abelian category, then there is an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker gf \longrightarrow \ker g \longrightarrow \operatorname{coker} f \longrightarrow \operatorname{coker} gf \longrightarrow \operatorname{coker} g \longrightarrow 0$$

This looks an awful lot like the conclusion to the snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "1"] \arrow[d, "f"] & A' \arrow[r, "f"] \arrow[d, "gf"] & A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A \arrow[r, "g"'] & A'' \arrow[r, "1"'] & A'' \arrow[r] & 0
\end{tikzcd} \qquad (**)$$

except of course that rows are never exact (with some trivial exceptions). So it is natural to ask the question of what is required of a commutative diagram of the form (*) to force the conclusion of the snake lemma. We do not give a complete necessary and sufficient condition, but we do find a sufficient condition that includes both the cases $(*)$ and $(**)$. We do not know if the condition we find is necessary; my best guess is that it is not.