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Feedback for linearly distributive categories: traces and fixpoints

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Dedicated to Bill Lawvere to mark the occasion of his 60th birthday

Abstract

In the present paper, we develop the notion of a trace operator on a linearly distributive category, which amounts to essentially working within a subcategory (the *core*) which has the same sort of "type degeneracy" as a compact closed category. We also explore the possibility that an object may have several trace structures, introducing a notion of *compatibility* in this case. We show that if we restrict to compatible classes of trace operators, an object may have at most one trace structure (for a given tensor structure). We give a linearly distributive version of the "geometry of interaction" construction, and verify that we obtain a linearly distributive category in which traces become canonical. We explore the relationship between our notions of trace and fixpoint operators, and show that an object admits a fixpoint combinator precisely when it admits a trace and is a cocommutative comonoid. This generalises an observation of Hyland and Hasegawa. © 2000 Elsevier Science B.V. All rights reserved.

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Introduction

In order to understand the computational aspects of the cut elimination process, in particular with respect to linear logic, Girard [13] makes a distinction between the denotational semantics of a logic and a quite different idea, which he calls "geometry of interaction". Girard observes that the denotational semantics of a logic measures the outcome of normalization: every proof is equivalent to a cut-free proof. What this viewpoint fails to capture is the *dynamics* of cut-elimination. In [14], Girard proposed a specific model of geometry of interaction: proofs are interpreted as operators on a Hilbert space, and cut-elimination is achieved by the iteration of a single operator. The dynamics were then captured by an "execution formula" which described the iteration required to normalize the terms.

In [3], Abramsky and Jagadeesan proposed a reformulation of Girard's ideas which was much more categorical in flavour. Rather than using Hilbert spaces, the Abramsky–Jagadeesan version of the "geometry of interaction" construction worked on categories of domains and produced a model of linear logic. In this new construction the execution formula relied on the presence of fixed points and was motivated by the semantics of feedback in dataflow networks.

In [20] Joyal et al. introduced the notion of a traced monoidal category. The trace was designed to model such constructs as braid closure, feedback, and, of course, the trace operator on finite-dimensional Hilbert spaces. They proved that every compact closed category has a canonical trace and that every traced monoidal category embeds into a compact closed category while making the trace canonical. This last result was obtained by a construction which functorially assigned to a traced monoidal category a compact closed category: their construction was essentially the same as the Abramsky–Jagadeesan "geometry of interaction" construction [1].

Any doubt about the relationship between these two constructions was removed when Hyland and Hasegawa [15] independently observed that a category with a traced product is precisely the same thing as a category with fixed points. Thus, the Joyal–Street–Verity construction was an abstract reformulation of the Abramsky–Jagadeesan "geometry of interaction" construction.

In retrospect, Girard's original construction can also be seen as exploiting the presence of a trace. Thus, it is reasonable to view a "geometry of interaction" semantics as being one given by an execution formula determined by a trace. (This has recently been made precise by Hines [17].) Of course, this use of a trace to obtain a dynamical semantics for linear logic can now be seen to have an important side-effect: the codomain of the interpretation will be a compact closed category. This should be contrasted with the standard denotational semantics for linear logic which is an interpretation into a $*$-autonomous category.

In this way, Girard's rather concrete desire to understand the dynamic process of cut-elimination gives way to the development of an abstract understanding of trace combinators and their relation to fixed points. This understanding has wider ramifications than its original application to linear logic. For example, the fact that there are two apparently completely different semantic denotations, domain theoretic and iteration theoretic, of imperative programming constructs such as the "while loop" can now be simply explained: what is needed to interpret these constructs is a trace and both theories supply settings which are traced. The theory of traces opens the door to a better understanding of the various forms of feedback which occur in all walks of mathematical life from matrix traces to recursive equations.

The purpose of this paper is to develop the theory of trace (and fixpoint) combinators in the linearly distributive setting. We take a local view of the trace combinator: rather than assuming that a trace is available at every object, we consider the effect of particular objects having a trace. This allows us to separate the concerns of compatibility (Section 3), which arise when tracing is possible at multiple objects, from the mere presence of a trace (Section 2).

A trace is a special feedback combinator: it is a combinator or functional from maps to maps: thus, given a map $U \otimes A \xrightarrow{f} U \oplus B$ it delivers a map $A \xrightarrow{\mathrm{tr}(f)} B$. We start by regarding U as a constant, so that feedback only need be available at one object. While feedback is often available at all objects, there are many examples in which this is not the case. For example, in the category of finite posets a notion of trace (on the product) can be provided by using least fixed points. However, not every object in this category will have a trace, since the least fixed point construction requires a least element.

It should also be pointed out that a category, or indeed an object, can have more than one trace structure. Thus, in finite posets the greatest fixed point operator is also available as a basis for the construction of a trace. Of course, these two different trace structures are not compatible (in the sense of Section 3), and indeed we shall show that two compatible trace structures must coincide. (Similar observations have been noted in a slightly different setting by Simpson [22].)

There are many different notions of feedback, and one might remark that they need not all satisfy the axioms demanded of a trace in the sense of [20]. For example, the main (indeed, only) non-structural trace axiom is "yanking" (Section 2), which says that feedback on the "switch" map is the identity. Yanking is definitely not satisfied by the usual notion of feedback in "stream processing", where one delays the output until the next time step when one uses it as input: for streams, feedback on the switch is delay.

It should also be mentioned that a notion of feedback which is specific to certain maps (not just objects) is also possible, see [2]. There the authors introduce the notion of a trace ideal. The motivating example is in the category of Hilbert spaces: while many morphisms do not have a trace, within each $\mathrm{Hom}(\mathscr{H}, \mathscr{H})$ there is a subspace of morphisms which can be traced. This subspace forms a two-sided ideal which is closely related to the ideal of Hilbert–Schmidt maps. In Remarks 8 and 16 we discuss how these ideas can be generalised to this setting.

The denotational semantics of (multiplicative) linear logic is essentially given by $*$-autonomous categories. From this perspective, compact closed categories are slightly degenerate, since they correspond to models of linear logic in which the two multiplicative connectives are canonically isomorphic. However, compact closed categories are the natural doctrine to model the geometry of interaction semantics; in other words, the distinction between $*$-autonomous and compact closed categories is essentially equivalent to the distinction between denotational semantics and geometry of interaction semantics for linear logic. As indicated above, in the present paper we propose a construction which attempts to bridge this gap, namely the notion of a trace operator on a $*$-autonomous category, or more generally on a linearly distributive category. Even though $*$-autonomous categories make up the basic ingredient of categorical models of linear logic, it is quite productive to ignore the closed structure entirely and instead focus on the interaction between the tensor product and its dual cotensor, *par*. This was one of the motivations of the latter two authors in introducing linearly distributive categories. In a sequence of papers [6,7,9,11,12], it has been amply demonstrated that once one understands the linearly distributive structure, the extension of crucial structural results to $*$-autonomy is straightforward. These results are achieved by introducing a graph-theoretic language for specifying morphisms which is inspired by proof nets. It should be thought of as a logical version of the Joyal–Street geometry of tensor calculus [19].

Furthermore, the general "geometry of interaction" construction (Section 4) completes a category by adding "complements" to make the traces canonical. The construction, however, is pointless if all the complements are already present: thus, it is crucial to start in a setting which can lack complements. Linearly distributive categories, being the notion of $*$-autonomous categories without complementation, are therefore a natural starting point from which to consider such a construction.

Significantly, to make sense of a trace combinator in the linearly distributive setting it is necessary to suppose the MIX rule holds. In a MIX category, the very fact that an object has a trace immediately forces the object to be in the "core" (Section 1.2) where the distinction between tensor and par is lost. This reflects the fact that a geometry of interaction semantics necessarily lies in a compact closed category. We also explore the notion that an object may have several traced structures, and we introduce a notion of *compatibility* in this case, which turns out to be equivalent to the dinaturality of the trace operator, and so to the axiom "sliding" of [20]. For a given tensor structure, restricting to compatible classes of trace operators guarantees that an object may have at most one trace structure.

The link between fixpoint combinators and trace combinators can also be expressed in this setting (Section 5). We show that an object admits a fixpoint combinator precisely when it admits a trace and it is a cocommutative comonoid. This generalises the observation by Hyland and Hasegawa [15]

Finally, we repeat a frequent warning about terminology and notation from previous papers in this series. The reader will have already noticed that we have adopted the term "linearly distributive category" for what previously we have called "weakly distributive category", continuing the practice begun in [11]. More controversial perhaps is our insistence upon the use of $\oplus$ for "par", preferring $+$ for the coproduct "sum".

1. The core of a MIX category

1.1. Preliminaries

1.1.1. Linearly distributive categories

For the full definition of a linearly distributive category, we refer the reader to [7,9,10] (where the term "weakly distributive category" is used). Briefly, a linearly distributive category is a category with two tensors $\otimes, \oplus$ and two strength natural transformations, making each tensor strong (respectively costrong) with respect to the other. These strength transformations will be denoted by

$$\delta^L_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C,$$
$$\delta^R_R : (B \oplus C) \otimes A \to B \oplus (C \otimes A).$$

In this paper, we shall suppose these tensors are symmetric, and so we have two additional induced strength transformations:

$$\delta^L_R : A \otimes (B \oplus C) \to B \oplus (A \otimes C),$$
$$\delta^R_L : (B \oplus C) \otimes A \to (B \otimes A) \oplus C.$$

In this case, δ^R_R is induced from δ^L_L and need not be assumed as a primitive. These data must satisfy standard coherence conditions, discussed in [9]. These tensors must satisfy the usual conditions for a monoidal category, and in addition there are conditions for the "distributivities" above; in the symmetric case which is the main concern in the present paper, these essentially amount to requiring the following commutative diagrams:

$$\begin{tikzcd}
\top \otimes (A \oplus B) \arrow[d, "\delta^L_L"'] \arrow[dr, "u^L_\otimes"] & \\
(\top \otimes A) \oplus B \arrow[r, "u^L_\otimes \oplus 1_B"'] & A \oplus B
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes (C \oplus D) \arrow[r, "a_\otimes"] \arrow[dd, "\delta^L_L"'] & A \otimes (B \otimes (C \oplus D)) \arrow[d, "1_A \otimes \delta^L_L"] \\
{} \arrow[r, phantom, "="] & A \otimes ((B \otimes C) \oplus D) \arrow[d, "\delta^L_L"] \\
((A \otimes B) \otimes C) \oplus D \arrow[r, "a_\otimes \oplus 1_D"'] & (A \otimes (B \otimes C)) \oplus D
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \oplus B) \otimes (C \oplus D) \arrow[dl, "\delta^L_L"'] \arrow[dr, "\delta^R_R"] & \\
((A \oplus B) \otimes C) \oplus D \arrow[d, "\delta^R_R \oplus 1_D"'] & = & A \oplus (B \otimes (C \oplus D)) \arrow[d, "1_A \oplus \delta^L_L"] \\
(A \oplus (B \otimes C)) \oplus D \arrow[rr, "a_\otimes"'] & & A \oplus ((B \otimes C) \oplus D)
\end{tikzcd}$$

The third diagram above is the most controversial, as it fails to be true in distributive categories, and is why distributive categories cannot be linearly distributive (unless they are posetal, see [10]). However, it is a direct consequence of the logical interpretation of linearly distributive categories: it corresponds to a natural (and necessary) cut-elimination step. For in essence, ignoring associativity,

$$\delta_L^L;\delta_R^R\oplus 1=\cfrac{\cfrac{B,C\to B\otimes C\quad A\oplus B\to A,B}{A\oplus B,C\to A,B\otimes C}\quad C\oplus D\to C,D}{A\oplus B,C\oplus D\to A,B\otimes C,D}$$

and

$$\delta_R^R;1\oplus\delta_L^L=\cfrac{\cfrac{B,C\to B\otimes C\quad C\oplus D\to C,D}{B,C\oplus D\to B\otimes C,D}\quad A\oplus B\to A,B}{A\oplus B,C\oplus D\to A,B\otimes C,D}$$

and a standard permutation of the cuts would require these to be equivalent.

We shall see some examples below in Example 4.

1.1.2. Polycategorical composition and circuits

In [9] we showed that the linear distributivities are precisely what is necessary to model the *cut* rule for polycategories (or equivalently, for sequent calculus with multiple conclusions and multiple premises). In this paper it will be convenient to use the polycategorical cut rule, which we shall call "polycategorical composition"; the reader ought to consult [9] for further details, but the following example ought to make the notion clear. Suppose we have maps $C_1\otimes A\otimes C_2\xrightarrow{g}C_3$ and $D_1\xrightarrow{f}D_2\oplus A\oplus D_3$ (one may imagine the objects C_i, D_j are finite "sequences" of objects, i.e. finite tensors or pars of such sequences, as appropriate). We can construct $f\,\text{⨾}\,g\colon C_1\otimes D_1\otimes C_2\to D_2\oplus C_3\oplus D_3$ as follows (ignoring several instances of associativity for simplicity):

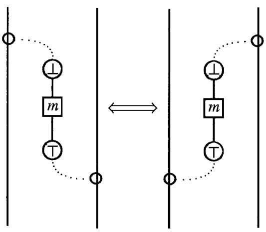
$$\begin{aligned}
f\,\text{⨾}\,g = C_1\otimes D_1\otimes C_2 &\xrightarrow{1\otimes f\otimes 1} C_1\otimes(D_2\oplus A\oplus D_3)\otimes C_2\\
&\xrightarrow{\delta_R^L\otimes 1}(D_2\oplus(C_1\otimes(A\oplus D_3)))\otimes C_2\\
&\xrightarrow{\delta_R^R} D_2\oplus((C_1\otimes(A\oplus D_3))\otimes C_2)\\
&\xrightarrow{1\oplus(\delta_L^L\otimes 1)} D_2\oplus(((C_1\otimes A)\oplus D_3)\otimes C_2)\\
&\xrightarrow{1\oplus\delta_L^R} D_2\oplus((C_1\otimes A\otimes C_2)\oplus D_3)\\
&\xrightarrow{1\oplus g\oplus 1} D_2\oplus C_3\oplus D_3
\end{aligned}$$

(where 1 represents identity morphisms). There are other ways of achieving this polycategorical composition, but they are equivalent under the coherence conditions imposed on linearly distributive categories.

Before leaving this subsection, we ought to remark on a distinction that must be made between the circuit diagrams we use and the categorical diagrams they are intended to illuminate. Circuits in fact correspond to morphisms in polycategories, not categories. One can get a categorical morphism by "tensoring" inputs and "par'ing" outputs, but unless this is systematically done throughout the circuit, there will be polycategorical elements remaining in the circuit. There is a direct translation between polycategorical morphisms and categorical ones; in [9] we showed that adding tensor and par to polycategories was conservative (in the categorical sense of an adjunction with a fully faithful unit), and that linearly distributive categories are equivalent to polycategories with tensor and par. But since we often place our discussions in the context of circuits, rather than using categorical conditions, the reader must keep the distinction clear. Circuits and categorical diagrams emphasize different aspects of the underlying structure; it is to be expected that in translating between them, certain features will gain or lose prominence. But we must be clear about the following: circuits are a precise means of discussing the categorical (as well as the polycategorical) structure of the categories we consider. There is a precise "term logic" for them, given in [7], which makes them more than just an analogy. Using this, the reader who wants to recast our language into categorical terms may do so. Our point, of course, is that the present presentation helps make the structure easier to understand. Circuits handle the various instances of tensorial strength that determine the structure of linearly distributive categories with particular elegance.

A piece of terminology: we refer to non-logical axioms (or generating morphisms) as "components"; one of the key results of [7] is that for purposes of determining the equivalence of circuits (i.e. the equality of maps), rules given in terms of components may in fact be used with arbitrary (sequentializable) subcircuits[4] playing the same role as components.

1.1.3. MIX categories



Next, we recall from [11] that a MIX category is a linearly distributive category with a morphism $m : \bot \to \top$, satisfying a simple coherence condition. Note that in a MIX category, there is a morphism (also denoted m) $A \otimes B \stackrel{m_{AB}}{\to} A \oplus B$ for any A, B, which essentially amounts to either of the equivalent nets at left. The coherence condition referred to above is just that these two canonical ways of constructing this map are equal. In circuits, this condition amounts to being able to "switch", or rewire, the unit thinning links that may be attached to the m component, as in the figure. In fact, in [11] we show that this condition need only be supposed to hold when the two wires have a unit type. We can strengthen the definition of mix: an isoMIX category is a MIX

[4] Unless we state otherwise, we shall use "subcircuits" to refer to sequentializable subgraphs, and "subgraphs" to refer to those that are not necessarily sequentializable.

category whose "mix" morphism $m : \bot \xrightarrow{\sim} \top$ is an isomorphism. (Note this does not force $A \otimes B \xrightarrow{m_{AB}} A \oplus B$ to be an isomorphism.) The isoMIX condition is essentially equivalent to having a biunit, and indeed simply forcing the units to be isomorphic also forces the mix condition. Thus a linearly distributive category in which $\top$ is isomorphic to $\bot$ is an isoMIX category.

In a symmetric linearly distributive category, a morphism $f : A \to B$ is said to be nuclear [12] if there are morphisms $\tau : \top \to C \oplus B$, $\gamma : A \otimes C \to \bot$ so that $(u^R_\otimes)^{-1}; 1 \otimes \tau; \delta^L_L; \gamma \oplus 1; u^L_\oplus = f$ (where u are appropriate unit isomorphisms). We say C (and τ, γ) "witnesses" the nuclearity of f. An object is nuclear if its identity map is nuclear; nuclear morphisms form a 2-sided ideal. The nucleus of a linearly distributive category is the full subcategory of its nuclear objects. In the symmetric case, this operation is idempotent (though not in the non-symmetric case).

We noted in [11] that a linearly distributive category is MIX if and only if its nucleus is. Related to this is the notion of complement: an object A of a linearly distributive category is said to be complemented if there is an object B and maps $\tau : \top \to B \oplus A$, $\gamma : A \otimes B \to \bot$ so that $(u^R_\otimes)^{-1}; 1 \otimes \tau; \delta^L_L; \gamma \oplus 1; u^L_\oplus = 1_A$ and $(u^L_\otimes)^{-1}; \tau \otimes 1; \delta^R_R; 1 \oplus \gamma; u^R_\oplus = 1_B$. This means that each object A, B is nuclear, and moreover each is a witness of the other's nuclearity. For a complemented object A, its complement is unique up to a unique isomorphism. If idempotents split, then nuclear objects must be complemented, so these two notions coincide.

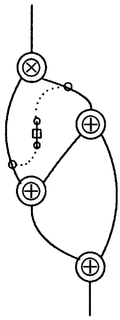
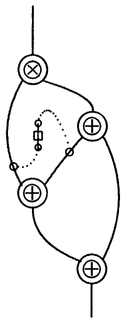
1.2. The core

Definition 1. Suppose $\mathbf{X}$ is a MIX category. We say an object U is in the *core of* $\mathbf{X}$ if the natural transformation $U \otimes \_ \xrightarrow{m_{U\_}} U \oplus \_$ is an isomorphism.

Lemma 2. *The following diagram commutes in a* MIX *category. So, if U is in the core of such a category, then the linear distributivity δ^L_L is an isomorphism (essentially corresponding to associativity).*

$$\begin{array}{ccc}
U \otimes (A \oplus B) & \xrightarrow{\delta^L_L} & (U \otimes A) \oplus B \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m \oplus 1} \\
U \oplus (A \oplus B) & \xrightarrow{a_\oplus^{-1}} & (U \oplus A) \oplus B
\end{array}$$

Proof. This is most simply shown by examining the proof circuits for the maps involved. Throughout this paper, we represent the "MIX-barbell" by . Note it has thinning links attached at either end; rewiring these is a key step in such proofs. In [7] we gave a Rewiring Theorem which showed that any rewiring past a subcircuit was valid; the reader ought to consult that paper for the full details.

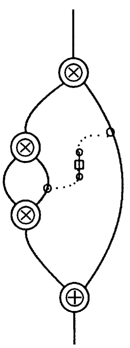
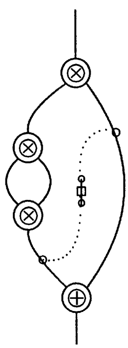
$m\,;a_{\otimes}^{-1} =$  $=$  $= \delta_L^L\,;m \oplus 1$ $\square$

Proposition 3. *If* $\mathbf{X}$ *is a* MIX *category and* U, V *are in the core of* $\mathbf{X}$, *then* $U \otimes V$, $U \oplus V$ *are also in the core. Moreover,* $U \otimes V$, $U \oplus V$ *are isomorphic. If* $\mathbf{X}$ *is iso*MIX, *then* $\top$, $\bot$ *are also in the core (so that for an iso*MIX *category, the core forms a full compact (i.e.* $\otimes \cong \oplus$*) linearly distributive sub-category of* $\mathbf{X}$*).*

Proof. Most of this is obvious; the only point that needs some elaboration is that $U \otimes V$ is in the core if U and V are. This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
(U \otimes V) \otimes A \arrow[r, "m"] \arrow[d, "a_{\otimes}"'] & (U \otimes V) \oplus A \\
U \otimes (V \otimes A) \arrow[r, "1 \otimes m"] & U \otimes (V \oplus A) \arrow[u, "\delta_L^L"']
\end{tikzcd}$$

The commutativity of the diagram follows from the following circuit equation:

$a_{\otimes}; 1 \otimes m\,;\delta_L^L =$  $=$  $= m$ $\square$

Example 4. There are a number of examples of MIX categories which have non-empty cores but in which the tensor and par cannot be identified.

(i) In any isoMIX category the core is non-empty since the unit is in the core. However, this may be all that is in the core. If the category has biproducts acting as tensor and par, then it is not hard to show that all (finite) biproducts of the unit will also be in the core. This may then be a non-trivial category.

An example of this phenomenon is given by the category **RTVec** of reflexive linear topological vector spaces [5,8,21], i.e. vector spaces equipped with a linear topology which are isomorphic to their double duals. In this category, all finite-dimensional vector spaces are in the core but infinite-dimensional vector spaces are not in the core.

(ii) As another example, consider the category of sup-lattices, where the objects are lattices with arbitrary suprema and the functions preserve these; this is a well-known $*$-autonomous category [4]. The tensor is determined as the adjoint to the function space construction (where the functions are given the pointwise ordering) and the tensor unit is the two-element boolean algebra 2. This unit is also the dualizing object. The elements of $A \multimap 2$ correspond to ideals which, being closed under suprema, are principal. These ideals are ordered pointwise as maps to 2 which, in fact, means that they are ordered by the reverse of inclusion. Thus, the "perp" of an object is the sup-lattice itself but with the reverse ordering.

As the unit and counit in this category coincide it is in fact an isoMIX category. What is its core? First observe that sup-lattices have biproducts: thus the core contains at least all the biproducts of 2 (which are the finite boolean algebras). But clearly it contains more.

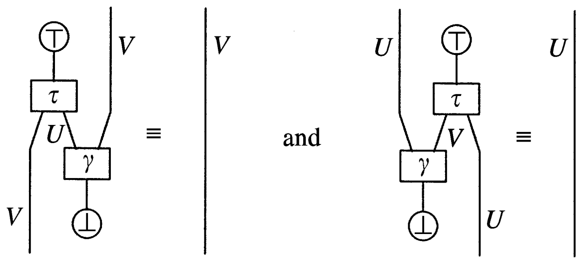
sup-lattice A is nuclear in case the function space can be expressed as $A \multimap B = (A \multimap 2) \otimes B$: in [16] these are shown to be the completely distributive lattices. However, since we are in a $*$-autonomous category this forces $A \multimap 2$ to be in the core for any nuclear A, and thus by Proposition 5 A must be in the core. The converse is also obviously true: that is core objects are nuclear. This means that objects in the core — which we shall see are also those objects which can be traced — in the category of sup-lattices are exactly the completely distributive lattices.

(iii) Any symmetric monoidal category $\mathbf{X}$ is an isoMIX category in which every object is in the core. The finite bicompletion of $\mathbf{X}$ is an isoMIX linearly distributive category $\Lambda(\mathbf{X})$, see [18]; its core includes $\mathbf{X}$ but is not the whole category.

(iv) In a symmetric monoidal category an object V is said to have a tensor inverse when there is an object V' such that $V \otimes V'$ is the unit (and certain coherence diagrams hold, see [9]). Given such an object one can define a "par" as $A \oplus B = A \otimes V \otimes B$ which has unit V'. If there is a map $\hat{m} : \top \to V$ then this provides a MIX structure; notice that in this case, the mix map is the mate $m : V' \to \top$ obtained by tensoring $\hat{m}$ with V'. An object is V-invariant in case $\hat{m} \otimes 1 : \top \otimes A \to V \otimes A$ is an isomorphism. Clearly V-invariant objects are in the core.

Proposition 5. *If U is in the core of $\mathbf{X}$ and V is a complement of U, then V is in the core of $\mathbf{X}$.*

Proof. Recall that $\langle U, V\rangle$ being a complement pair means that we have the following components and equalities:



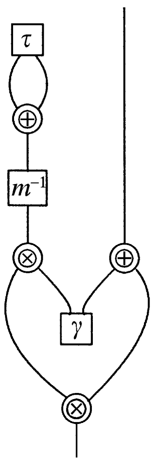
The inverse to $V \otimes A \xrightarrow{m} V \oplus A$ is

$$V \oplus A \xrightarrow{u} \top \otimes (V \oplus A) \xrightarrow{\tau \otimes 1} (V \oplus U) \otimes (V \oplus A)$$

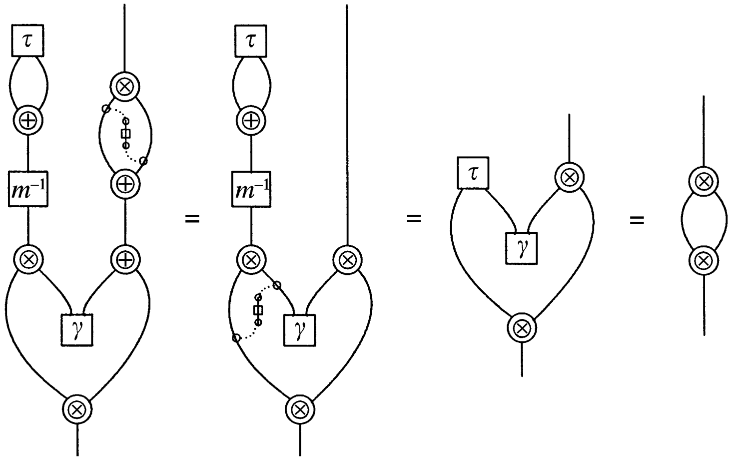
$$\xrightarrow{m^{-1} \otimes 1} (V \otimes U) \otimes (V \oplus A) \xrightarrow{a;\delta_L^L} V \otimes ((U \otimes V) \oplus A)$$

$$\xrightarrow{1 \otimes (\gamma \oplus 1)} V \otimes (\perp \oplus A) \xrightarrow{1 \otimes u} V \otimes A,$$

where m^{-1} is the inverse to $V \otimes U \to V \oplus U$ which exists since U is in the core. As a proof circuit, this map is the following. Note that to simplify these circuits, we shall drop the grounded unit nodes which are attached to the τ and γ nodes, writing, for example, the τ node without any input wires, and dually for γ:

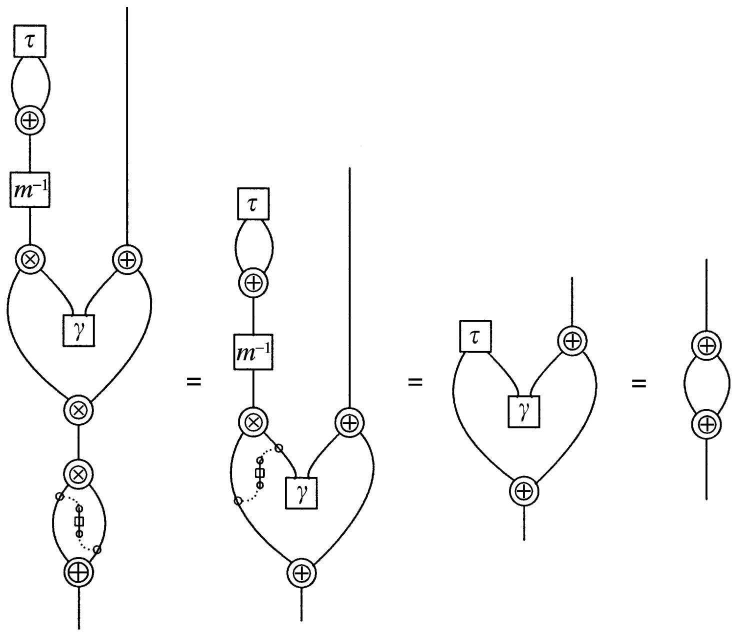


To see this is the required inverse amounts to some circuit rewrites. First, we precompose with m_{VA}, and show this is equivalent to the identity on $V \otimes A$:



which is the expanded normal form of the identity on $V \otimes A$. Note the rewirings around subcircuits, and the use of the equivalence $m^{-1}; m = 1$ in the second rewrite.

Next, we postcompose with m_{VA}, and show this is equivalent to the identity on $V \oplus A$, just as above:



which is the expanded normal form of the identity on $V \oplus A$. $\square$

2. Traced objects

We begin with a definition based upon the similarly named notion of Joyal et al. [20].

Definition 6. Suppose **X** is a MIX category, U an object of **X**. We say U has a *trace* if there is a family of functions $\mathrm{tr}_U^{AB}: \boldsymbol{X}(U \otimes A, U \oplus B) \to \boldsymbol{X}(A,B)$ satisfying the following axioms:

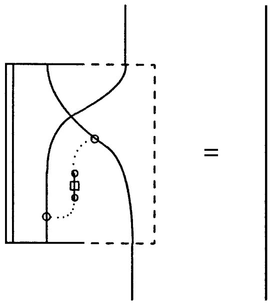
$$\begin{array}{ll} \text{Yanking}: & \mathrm{tr}_U^{UU}(c_\otimes; m_{UU}) = 1_U = \mathrm{tr}_U^{UU}(m_{UU}; c_\oplus) \\ \text{Tightening}: & \mathrm{tr}(g \fatsemi f \fatsemi h) = g \fatsemi \mathrm{tr}(f) \fatsemi h \\ \text{Superposing}: & \mathrm{tr}(f \otimes C) = \mathrm{tr}(f) \otimes C \\ & \mathrm{tr}(f \oplus C) = \mathrm{tr}(f) \oplus C \end{array}$$

We must clarify the notation. Above, $c_\otimes$ and $c_\oplus$ are the "twist" maps (which exist since we are assuming $\otimes$ and $\oplus$ are symmetric). First note that $c; m = m; c : U \otimes U \to U \oplus U$; these are the "twisted" versions of m. Then the meaning of "yanking" is clear: these are both sent to the identity on U under the trace operator. For "tightening", we suppose given morphisms (again we ignore instances of associativity where the meaning is clear, and will continue to do so when appropriate) $f: U \otimes A \otimes B \to U \oplus X \oplus Y$, $g: D \to B \oplus C$, $h: Y \otimes Z \to W$; here A, D, and Z may be thought of as arbitrary finite strings of (i.e. tensors of) objects, and X, C, and W may be thought of as arbitrary finite strings of (i.e. pars of) objects. The notation "$\fatsemi$" refers to the evident polycategorical composition discussed in the previous section, so the resultant equation is between maps $A \otimes D \otimes Z \to X \oplus W \oplus C$. Finally, for "superposing", we suppose f as for tightening, and then we mean that the trace of the map $U \otimes A \otimes B \otimes C \xrightarrow{f \otimes 1} (U \oplus X \oplus Y) \otimes C \xrightarrow{\delta'} U \oplus (X \oplus (Y \otimes C))$ is the map $A \otimes B \otimes C \xrightarrow{\mathrm{tr}(f) \otimes 1} (X \oplus Y) \otimes C \xrightarrow{\delta} X \oplus (Y \otimes C)$, and similarly for $\oplus$. Here the δ's are given by the evident linear distributivities. We shall present these axioms as circuit rewrites shortly.

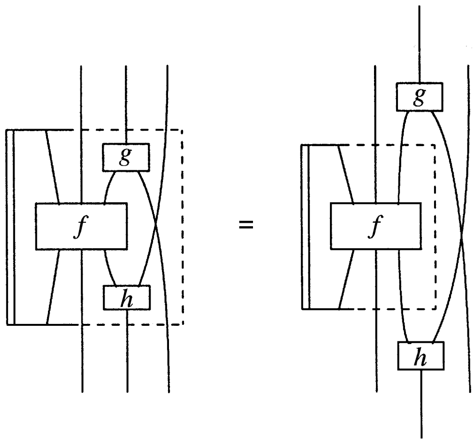
The names for these axioms are those of the corresponding axioms in [20]. There are two axioms in [20] we have omitted: "sliding", for which we shall soon have a replacement, and "vanishing", which will be a definition, not an axiom, in our treatment. Superposing is both more and less general here than in [20]: their version is only given in terms of the tensor, whereas we must also include a version for the par. Their version involves another map g, so we have the equation $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \otimes g$ and similarly for $\oplus$, but it is a simple exercise that this general version is a consequence of our version plus tightening (at least in the present context).

To present these axioms as proof circuits, we need a notation for the trace operator. We shall use a box notation similar to that used in [11], viz. we shall box the circuit corresponding to the map $U \otimes A \xrightarrow{f} U \oplus B$, anchoring the U wires to the box, leaving only the A, B wires, which will represent the map $A \to B$. When we wish to consider different trace operators, (for example, operators for different traced objects), we shall "decorate" the box with suitable labels. So, with these remarks to guide the reader, we present the three axioms for a trace operator in terms of proof circuits.

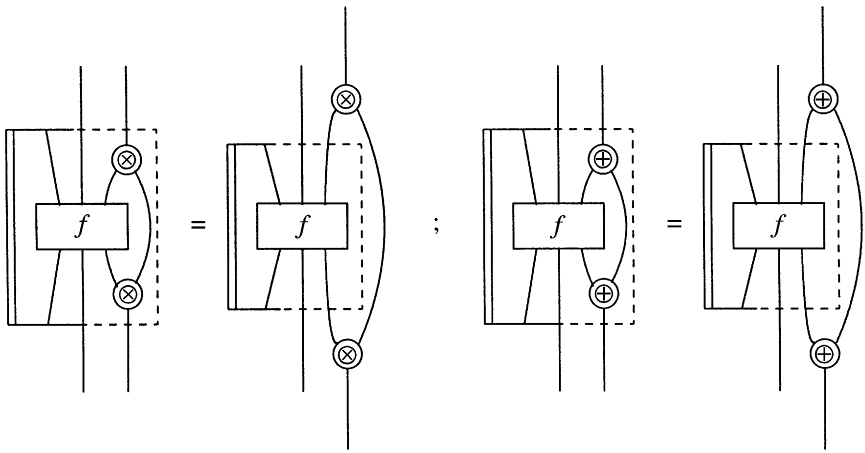
Yanking:



Tightening:



Superposing:



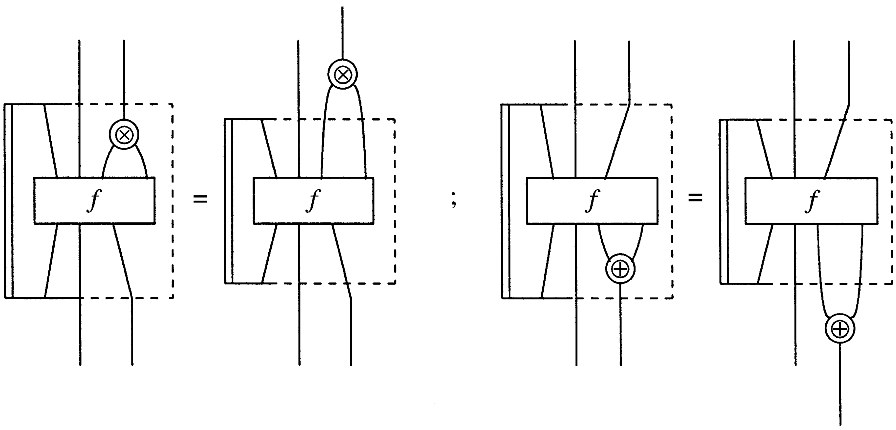
Remark 7 (*What the axioms mean*)**.** There is really only one "non-structural" axiom in the definition of a trace operator, viz. yanking. Tightening and superposing are "scope change" rules, analogous to those we had for the linear implication in [11].

Tightening is necessary in order to make the trace operator a strong combinator, and superposing is necessary for the category – polycategory translation that underlies all operations on linearly distributive categories. We shall consider these two axioms more closely in order to elucidate this structure.

First, it is standard to identify tightening as the requirement that $\mathrm{tr}_U^{A,B}$ is a natural transformation in A, B. The tightening axiom we have is a polycategorical generalisation of this, since we have allowed g, h to be "polymorphisms".[5] This amounts to adding a measure of tensorial strength to the situation; to explain this in detail would require a digression to introduce the notion of a "strong combinator": from this point of view, the trace operator acts on endomorphisms of U, and the strength allows the smooth handling of the parameters A and B. (An account of strong combinators in the cartesian case can be found in the thesis of Vesely [23] — the generalisation to the present context is fairly straightforward, but is not necessary for our purposes.)

Next we consider the axiom of superposing. Note that in the circuit rewrites for superposing, as given above, pulling the lower $\otimes$ node out of the trace box may be done with our "polycategorical" version of tightening, as may pulling out the upper $\oplus$ node, since these are subcircuits. So the essential content of this axiom involves pulling out the other nodes, which are not subcircuits. These nodes are "switching", in the terminology of proof nets, and their usual role is (for $\otimes$) to make two input wires into a single tensored input wire, and dually for the par, making two output wires a single par'ed output wire. In other words, they serve to translate between a polycategorical circuit, which has multiple in/output wires, and a categorical circuit, which has exactly one input and one output wire. Since these are *not* subcircuits (they do not correspond to morphisms), tightening does not apply to them; superposing then essentially amounts to allowing these moves, and so can be given in the following equivalent form.

Superposing (ii):



[5] We refer to "polymorphisms" to mean morphisms in a polycategory. As we pointed out in [9], any linearly distributive category may be regarded as a polycategory, and our circuits make such a viewpoint very natural. Thinking of a morphism as a component box in circuit notation, a morphism has one input and one output wire, whereas a polymorphism has many inputs and outputs.

This form of superposing has the feature that if we try to give a categorical version, then since we use the very nodes that are being moved in and out of the scope of the trace boxes, these equations end up being identities. So in effect, superposing (in our context) amounts to enabling the polycategorical – categorical translation. (A similar effect was noted in the linear implication scope boxes of [11].)

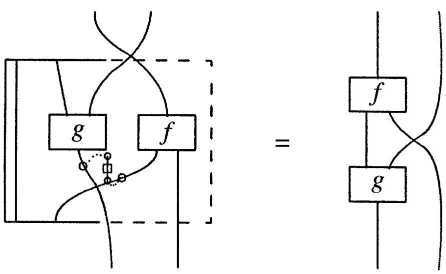
The point of these remarks is to underline the qualitative difference between the tightening and superposing requirements, and yanking: tightening and superposing are structural axioms which express no more than the strength of the combinator; yanking, however, is a distinct requirement — which makes the operator a trace rather than a general feedback combinator.

Remark 8 (*Trace ideals*). In [2], the authors introduce the notion of a trace ideal, a notion which models the fact that in (for example) Hilbert spaces one has many maps without a trace, in particular identity morphisms generally do not have traces. We can model this phenomenon in our setting by allowing tr_U to be a partial operator. The axioms which account for the bistrength, viz. tightening and superposing, will remain as before, but we must modify yanking, since there is no reason to suppose $c;m$ is in the domain of tr_U in general. In this setting, we would take the following variant of yanking, for $A \xrightarrow{f} U \oplus D$, $U \otimes C \xrightarrow{g} B$. If $g \otimes f \,{}_{9}^{\circ}\, m \,{}_{9}^{\circ}\, c$ is in the domain of $\mathrm{tr}_U^{C\otimes A, B\oplus D}$, then

$$\text{Generalised Yanking}: c \,{}_{9}^{\circ}\, \mathrm{tr}_U(g \otimes f \,{}_{9}^{\circ}\, m \,{}_{9}^{\circ}\, c) = f \,{}_{9}^{\circ}\, g.$$

As before, we may suppose A and B represent finite strings of objects.

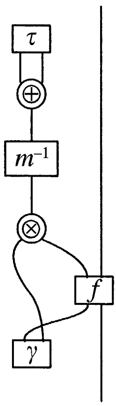
In circuit notation, this is the following rewrite:



Note that in the case of a global trace operator, this generalised version of yanking is a consequence of (ordinary) yanking and tightening. In the partial operator case, tightening is to be interpreted as requiring that (using the notation of Definition 6) if f is in the domain of tr, so is $g \,{}_{9}^{\circ}\, f \,{}_{9}^{\circ}\, h$, and moreover $\mathrm{tr}(g \,{}_{9}^{\circ}\, f \,{}_{9}^{\circ}\, h) = g \,{}_{9}^{\circ}\, \mathrm{tr}(f) \,{}_{9}^{\circ}\, h$. Likewise, superposing is to be interpreted as requiring that if f is in the domain of tr, so are $f \otimes C$, $f \oplus C$, and moreover $\mathrm{tr}(f \otimes C) = \mathrm{tr}(f) \otimes C$, $\mathrm{tr}(f \oplus C) = \mathrm{tr}(f) \oplus C$. With this revised definition, note that tightening guarantees that the domain of tr_U^{AB} (for any A, B, U) is a two-sided ideal.

Our first key result about traces is that in a MIX category, any complemented core object has a canonical trace operator. This fact plays the role in our theory that the canonical trace on a tortile monoidal category plays in the theory of [20].

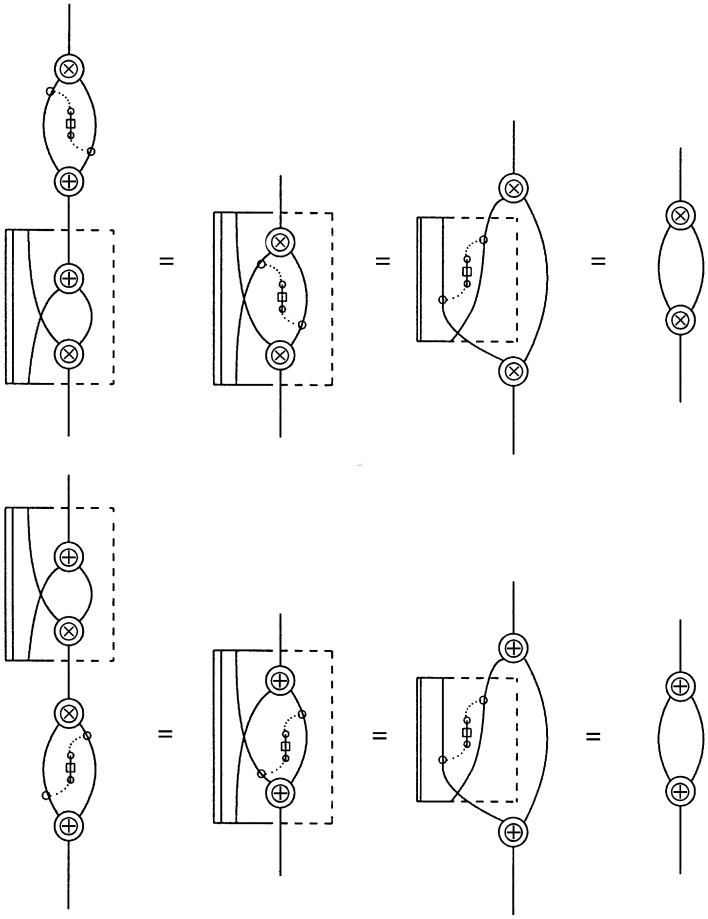
Proposition 9. *Suppose* **X** *is a* MIX *category*, U *a complemented object in the core of* **X**. *Then* U *has a trace, called the* complement trace, *defined as follows. For* $f: U \otimes A \to U \oplus B$, $\mathrm{tr}(f) = A \xrightarrow{u} A \otimes \top \xrightarrow{1\otimes\tau} A \otimes (V \oplus U) \xrightarrow{\delta'} V \oplus (U \otimes A) \xrightarrow{1\oplus f} V \oplus (U \oplus B) \xrightarrow{m^{-1}} V \otimes (U \oplus B) \xrightarrow{\delta''} (U \otimes V) \oplus B \xrightarrow{\gamma\oplus 1} \bot \oplus B \xrightarrow{u} B$. *This is given by the circuit below*:



Proof. There are three circuit diagrams to verify. The diagram for yanking is trivial: just rewire the MIX-barbell so it falls just below the m^{-1} node, and then the circuit reduces directly to the identity wire. For tightening and superposing, there is actually nothing to do: the circuits are the same on either side of the equations. □

Proposition 10. *If* U *is a traced object of a* MIX *category* **X**, *then* U *is in the core of* **X**.

Proof. The map inverse to $m: U \otimes A \to U \oplus A$ is the trace of the linear distributivity $\delta: U \otimes (U \oplus A) \to U \oplus (U \otimes A)$. The following rewrites show these are indeed inverse. The first shows that $m; \mathrm{tr}(\delta) = 1_{U\otimes A}$, the second shows that $\mathrm{tr}(\delta); m = 1_{U\oplus A}$. The main subtlety here is that in the first case we can rewire the thinning link at the bottom around the $\otimes$ link, and in the second case we must rewire the thinning link at the top around the $\oplus$ link. Switching the MIX-barbell to its mirror image is valid by the coherence condition for MIX categories:



□

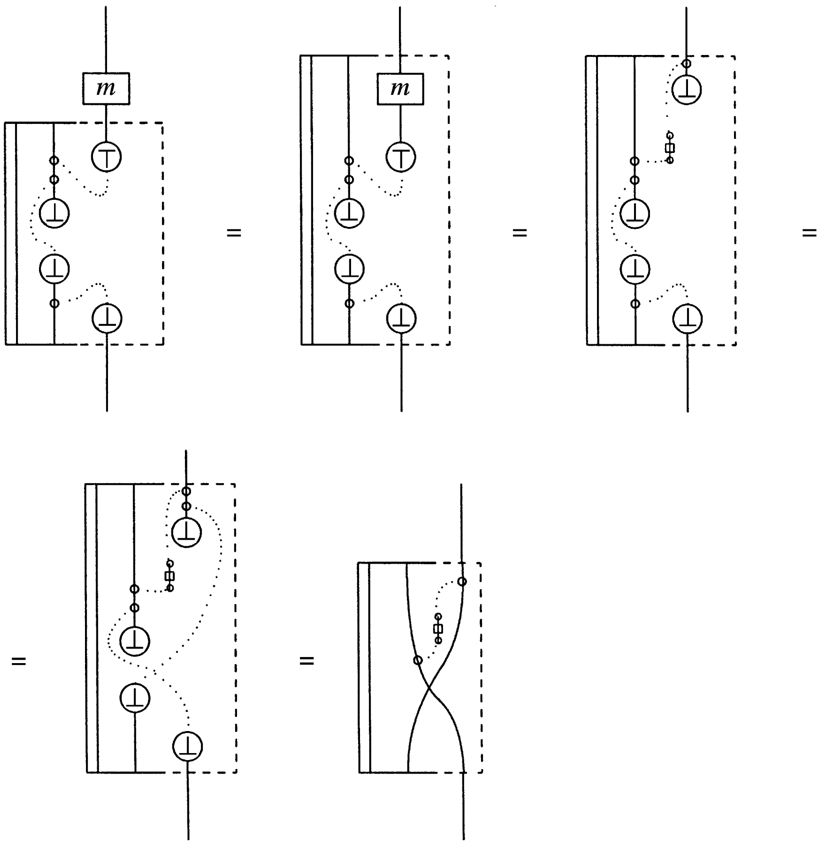
There is an analogous result for the tensor units. It is clear that in an isoMIX category, the (common) unit for the tensor and par is trivially traceable; however, the converse is also true.

Proposition 11. *Suppose* **X** *is a* MIX *category. If either* $\top$ *or* $\bot$ *has a trace, then* $\bot \cong \top$, *so that* **X** *is an iso*MIX *category.*

Note that this implies that making all objects traced would eliminate the distinction between the set-up of this paper, using linearly distributive categories, and that of [20], since we would then have an isoMIX category in which all objects were in the core, and linear distributivity essentially just becomes associativity.

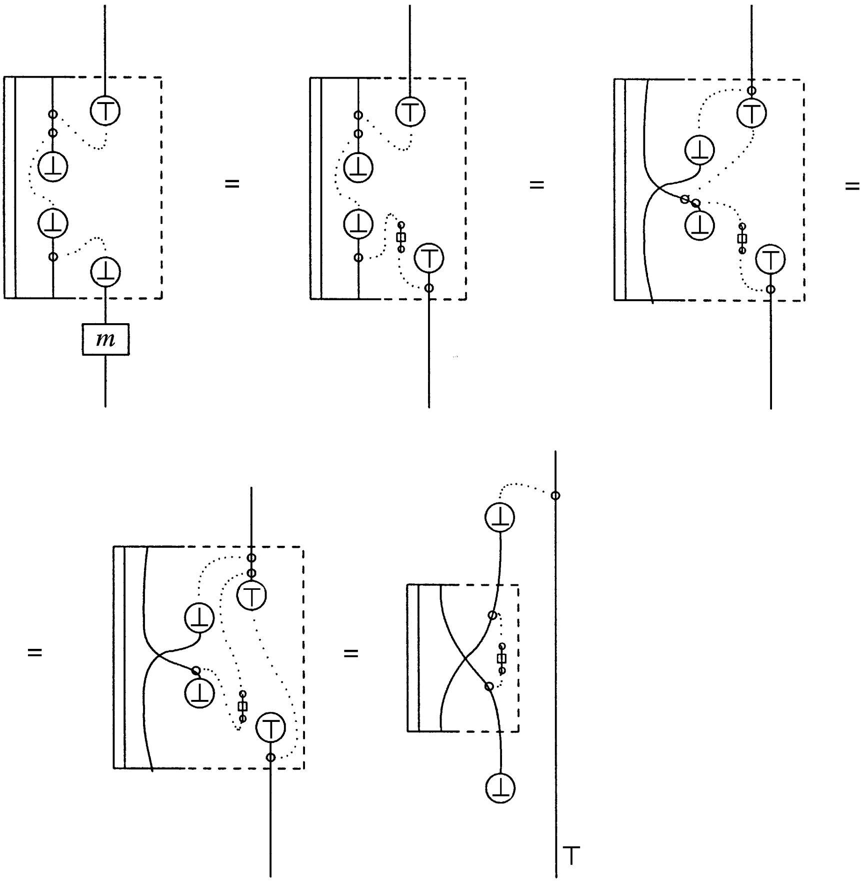
Proof. From an "abstract" point of view, this is obvious: if, say, $\top$ is traced, then it is in the core, so the functor $\top \otimes \_$ is isomorphic to the functor $\top \oplus \_$, and so $\top$ is a unit for the par. However, it may be of interest to see what the isomorphism is explicitly. We begin with the observation that if any object has a trace operator, then there is induced a morphism from $\top$ to $\perp$, viz. the trace of the morphism $U\otimes\top \xrightarrow{u_\otimes} U \xrightarrow{u_\oplus^{-1}} \perp\oplus U$. In general, there is no reason for this to be the inverse for $m:\perp\to\top$, but if U is either unit, then we can show that it is indeed the inverse. We shall consider the case $U=\perp$; the other case is similar. The key step is to note that if we have m on a wire, we can split the wire above and/or below the m using the unit expansion rewrites, which creates a MIX-barbell . Likewise we can split the unit wire which is attached to the trace box, again creating a thinning link. Then the rest of the proof involves rewiring the unit thinning links as necessary. (This can be somewhat subtle, and the order in which such rewirings is done can be vital, as we showed in [7]. A rewiring can alter the empires of other units, thus altering what other rewirings are possible. In this way, rewirings become possible that were blocked before.)

First we consider the composite $m;\mathrm{tr}(u;u^{-1})$:



which is the identity wire on $\perp$, by yanking. Note the step at the second equality, where we split the $\perp$ wire just above the m, creating a MIX-barbell, and the rewiring at the next step, where we rewired two thinning links, making it possible to join up the $\perp$ wires to give the last circuit.

Next, we consider the other composite; the steps are similar, although more rewiring is necessary.

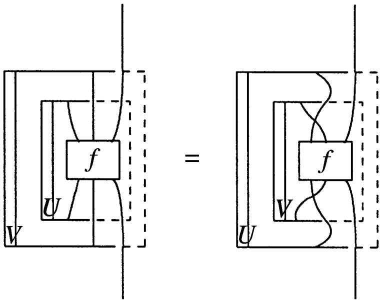


which, after yanking and unit reduction is the identity wire on $\top$. $\square$

One might ask whether an object U may have more than one trace operator. If we add a further condition, "compatibility", then in fact this is not possible, as we shall see in the next section.

3. Compatible traces

Definition 12. Suppose $\mathbf{X}$ is a MIX category, and U, V objects of $\mathbf{X}$ each with a trace operator, say tr_U, tr_V. These traces are called *compatible* if for any $f : U\otimes V\otimes A \to U\oplus V\oplus B$, $\mathrm{tr}_V(\mathrm{tr}_U(f)) = \mathrm{tr}_U(\mathrm{tr}_V(f'))$, where $f' = V\otimes U\otimes A \xrightarrow{c\otimes 1} U\otimes V\otimes A \xrightarrow{f} U\oplus V\oplus B \xrightarrow{c\oplus 1} V\oplus U\oplus B$. In circuits, this is the following equation:

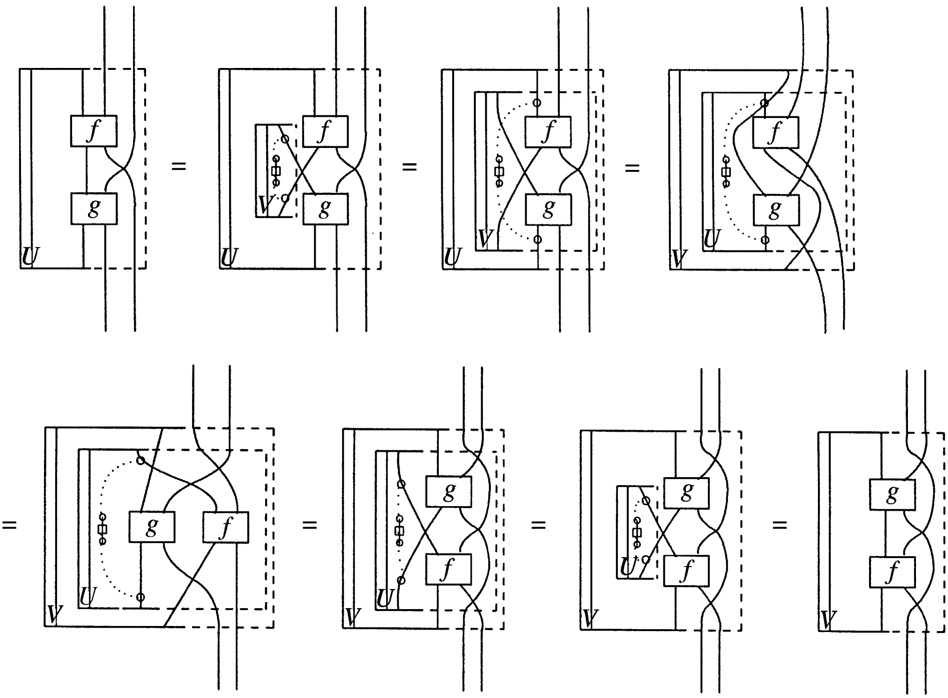


We shall say a trace operator is "self-compatible" if it is compatible with itself. This condition really ought to be considered part of the definition of a "good" notion of trace; we have separated it just to keep clear what notions depend on what conditions.

In [20] there is an axiom "sliding" which corresponds to the dinaturality in the variable U of the family tr_U. The equivalent equation in our context is the following consequence of compatibility.

Proposition 13. *Suppose* U, V *are objects of a* MIX *category with compatible trace operators* tr_U, tr_V. *Suppose* $f : U \otimes X \to V \oplus Y$ *and* $g : V \otimes A \to U \oplus B$; *let* $f \,{}_{9}^{\circ}\, g : U \otimes X \otimes A \to U \oplus B \oplus Y$ *and* $g \,{}_{9}^{\circ}\, f : V \otimes X \otimes A \to V \oplus B \oplus Y$ *be the evident "polycategorical compositions". Then* $\mathrm{tr}_U(f \,{}_{9}^{\circ}\, g) = \mathrm{tr}_V(g \,{}_{9}^{\circ}\, f)$.

Proof.



□

As a corollary, taking $U = V$, $A = \top$, $B = \bot$, and $g = 1$, we obtain the result we promised in the last section.

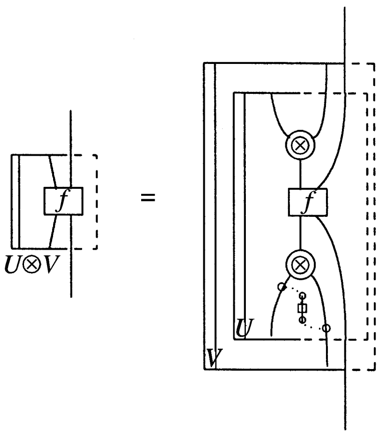
Corollary 14. *Any two compatible traces are equal.*

There remains an axiom of [20] that we have not yet considered, viz. "vanishing", which allows the trace of a tensor to be given in terms of the traces of the components of the tensor. (The nullary case of vanishing, viz. trace on the tensor unit is identity, is either trivially true, when $\top \cong \bot$, i.e. in the isoMIX case, or does not even type properly, so we shall not deal further with that case of the vanishing axiom.) Now, it is fairly easy to define a trace operator on $U \otimes V$ given traces on U and V: indeed there are two natural candidates, viz. for $f : (U \otimes V) \otimes A \to (U \otimes V) \oplus B$, $\mathrm{tr}_V(\mathrm{tr}_U(f \,\mathrm{;}\, m))$ and its "twisted" variant $\mathrm{tr}_U(\mathrm{tr}_V(c \,\mathrm{;}\, f \,\mathrm{;}\, m \,\mathrm{;}\, c))$. It is easy to see that these operators are both compatible with any operators which are compatible with tr_U and tr_V, but unless these last two are compatible with each other, there is no reason for them to be equal. So, although a slightly greater generality is possible, it seems most natural to define traces on tensors when the individual traces are compatible. This then leads us to the following definition/proposition.

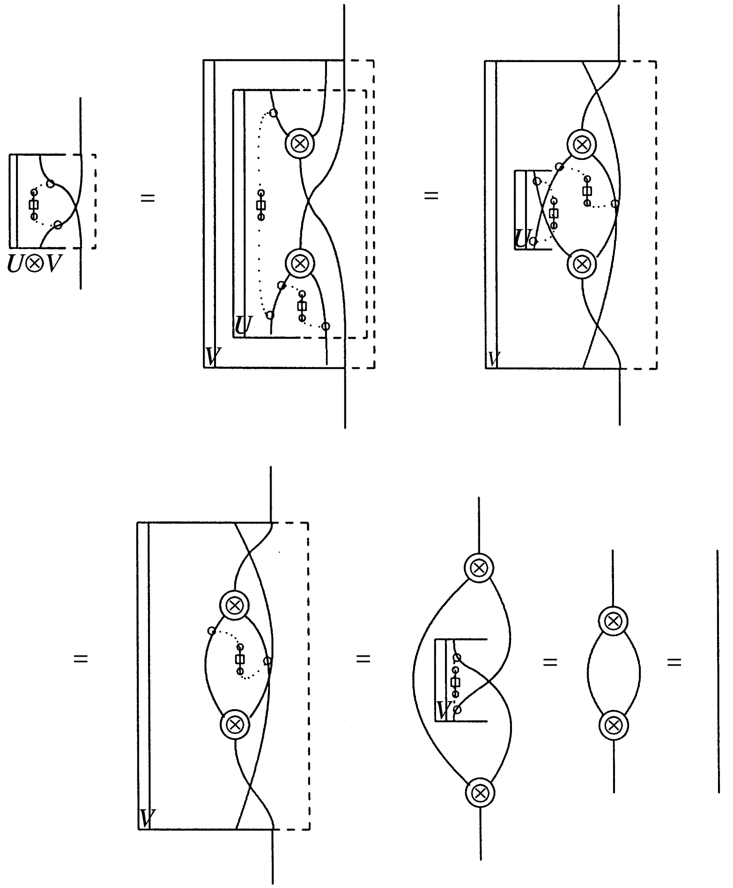
Proposition 15. *Suppose U, V are objects of a* MIX *category with compatible, self-compatible trace operators* tr_U, tr_V. *Then there is a canonical trace operator on $U \otimes V$ (as defined above) which is compatible with* tr_U, tr_V, *and in general is compatible with any trace which is compatible with* tr_U, tr_V. *In particular, it is self-compatible.*

Note that Corollary 14 shows that there can be at most one trace operator on $U \otimes V$ with these properties, since any two such traces must be compatible.

Proof. The trace on $U \otimes V$ has been defined above; as a circuit this is the following (we leave to the reader the construction of the "twisted" variant).



To show this is indeed a trace is fairly straightforward — the only point that requires some effort is to verify yanking (tightening and superposing are trivial). This we do with the rewrites below. The rest of the proposition, the statements about compatibility, is trivial.



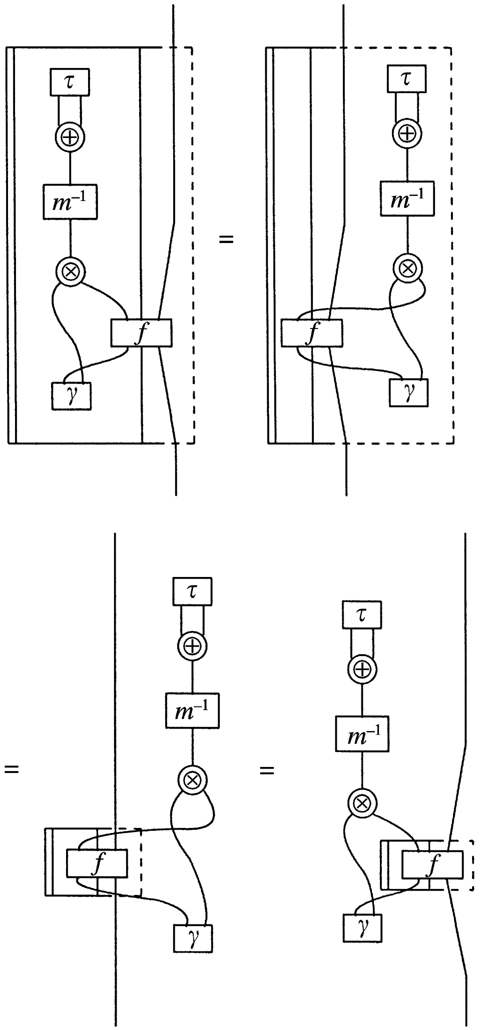
□

Note that if we take $U \otimes V$ (or the isomorphic $U \oplus V$) as the trace object, then (by Proposition 15) Definition 12 just amounts to the naturality condition for $\mathrm{tr}_{U\otimes V}$. So in essence, dinaturality in U = sliding = compatibility.

Remark 16 (*Trace ideals, continued*)**.** Continuing the ideas of Remark 8, we can extend the notion of compatible trace operators to the partial operator case. Here in Definition 12, we must interpret the equation $\mathrm{tr}_V(\mathrm{tr}_U(f)) = \mathrm{tr}_U(\mathrm{tr}_V(f'))$ in the sense that if one side is defined, so is the other, and the equality holds. Then in this case, it is easy to see that Proposition 13 holds, in a similar sense, that one trace is defined if the other is, and the equality holds. The proof is even somewhat simpler, as generalised yanking allows several steps to be combined in one. So, if we suppose that the partial trace operators are pairwise compatible, we have the key properties of a trace ideal, in the sense of [2], viz. that the domains are ideals, that the traces satisfy sliding, and that trace maps are closed under $\otimes$ and $\oplus$.

Proposition 17. *If U is complemented, then any trace on U is compatible with the complement trace.*

Proof.



□

Corollary 18. *If U is complemented, then any trace on U is equal to the complement trace.*

4. The geometry of interaction construction for MIX categories

In [20] a construction, originally given by Abramsky and Jagadeesan [3], is given of a tortile monoidal category $\mathrm{Int}\,\mathscr{V}$ from a traced monoidal category $\mathscr{V}$, together with a full and faithful embedding $N : \mathscr{V} \to \mathrm{Int}\,\mathscr{V}$. The point about this is that this construction provides a complement to a traced object. In this section we propose to give the analogous construction in the present setting; again, our approach is somewhat more "local", in that we shall start with a set U of compatibly traced objects in a MIX category $\mathbf{X}$, and fully and faithfully embed the category $\mathbf{X}$ into a MIX category $\mathbf{X}[\mathsf{U}]$ so that the image of U lies in the nucleus of $\mathbf{X}[\mathsf{U}]$.

Definition 19. Suppose $\mathbf{X}$ is a MIX category, U a set of pairwise compatibly traced objects of $\mathbf{X}$. The category $\mathbf{X}[\mathsf{U}]$ is defined as follows. An object is an ordered tuple $([U_1, U_2, \ldots, U_n], A)$, where the (possibly empty) sequence $[U_1, U_2, \ldots, U_n]$ consists of objects of U, and A is an arbitrary object of $\mathbf{X}$. A morphism

$$f : ([U_1, U_2, \ldots, U_n], A) \to ([V_1, V_2, \ldots, V_m], B)$$

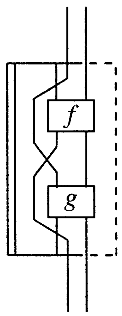
of $\mathbf{X}[\mathsf{U}]$ is a morphism

$$f : V_1 \otimes V_2 \otimes \cdots \otimes V_m \otimes A \to U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus B$$

of $\mathbf{X}$.

$\mathbf{X}[\mathsf{U}]$ is a category: the identity morphism for $([U_1, U_2, \ldots, U_n], A)$ is given by the MIX isomorphism (appropriately extended to many types) $m : U_1 \otimes \cdots \otimes U_n \otimes A \to U_1 \oplus \cdots \oplus U_n \oplus A$. (As a circuit, this is a set of parallel wires each joined to its neighbour by a MIX-barbell.) Composition is defined using the trace operators. Given $f : ([U_1, U_2, \ldots, U_n], A) \to ([V_1, V_2, \ldots, V_m], B)$, viz. $f : V_1 \otimes V_2 \otimes \cdots \otimes V_m \otimes A \to U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus B$, and $g : ([V_1, V_2, \ldots, V_m], B) \to ([W_1, W_2, \ldots, W_k], C)$, viz. $g : W_1 \otimes W_2 \otimes \cdots \otimes W_k \otimes B \to V_1 \oplus V_2 \oplus \cdots \oplus V_m \oplus C$, the composite $f;g$ in $\mathbf{X}[\mathsf{U}]$ is $\mathrm{tr}_{V_n}(\cdots \mathrm{tr}_{V_1}(c\,;f\,;c\,;g\,;c)\cdots)$.

Here, by c we mean sufficient uses of symmetry to bring the objects into the correct position for the trace to be applied. This is perhaps clearer in circuit notation; for example, assuming $m = n = k = 2$, the composition is the following circuit.



In general, imagine the left-hand wires (U, V, W) represent "ribbons" of wires, with the V ribbon caught in a series of trace boxes. So in effect, composition amounts to "poly-composition" (i.e. "cutting" the intermediate rightmost variables) and then tracing on the intermediate U-variables to eliminate them.

Remark 20. The key point to notice about this definition is the "contravariance" in the "first" variable $[U_1, U_2, \ldots, U_n]$. Such a sequence ought to be thought of as the tensor (or par — these are isomorphic since these objects all lie in the core of $\mathbf{X}$) of the individual U_i, and the pair $([U_1, U_2, \ldots, U_n], A)$ then ought to be thought of as the tensor (or par — again these are isomorphic) of the complement of $[U_1, U_2, \ldots, U_n]$ and A. This is why the notion of morphism "flips" the U's.

This definition is essentially just that given by [20] for Int $\mathscr{V}$; we use sequences of traced objects U_i instead of single objects, just so that we can simulate using the tensor unit as such a U (which is an option closed to us, since the units need not be traced) by using the empty sequence. We could in fact use sequences of length less than 2, if we supposed that the set U was closed under tensor (and so also closed under par, since U lies inside the core of $\mathbf{X}$). Although that is not necessary, in view of the next paragraph, we shall act as if that is in fact what we are doing.

In order to simplify the notation, we shall adopt the following convention. We shall represent an arbitrary morphism $f : V_1 \otimes V_2 \otimes \cdots \otimes V_m \otimes A \to U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus B$ by a map of the form $f : V \otimes A \to U \oplus B$, intending by this that V represents an arbitrary finite tensor of objects, and similarly U an arbitrary finite par of objects. This convention serves to avoid notational clutter, and makes it easier to see what is going on in the proofs. Generally a rigorous proof may be done by induction based on the pattern of the "two input – two output" case.

First, we must show that $\mathbf{X}[\mathsf{U}]$ actually is a category, and indeed, a linearly distributive category.

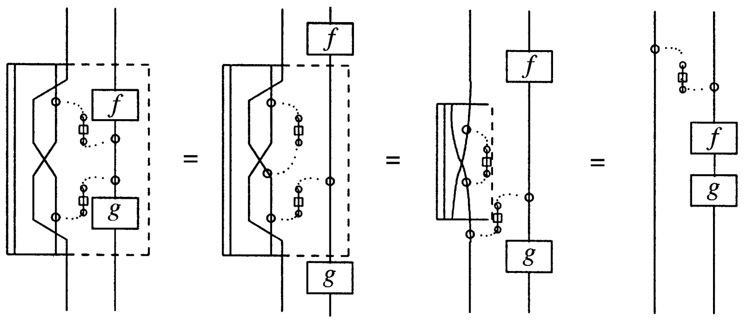
Theorem 21. $\mathbf{X}[\mathsf{U}]$ *is a linearly distributive category.*

Proof. To verify the categorical axioms for $\mathbf{X}[\mathsf{U}]$ is straightforward. The unit equations follow from yanking: for example, for $f : V \otimes A \to U \oplus B$, $1 \bar{;} f = \mathrm{tr}(c \mathbin{⨾} m \mathbin{⨾} c \mathbin{⨾} f \mathbin{⨾} c) = \mathrm{tr}(c \mathbin{⨾} c \mathbin{⨾} f \mathbin{⨾} m \mathbin{⨾} c) = f \mathbin{⨾} \mathrm{tr}(m \mathbin{⨾} c) = f \mathbin{⨾} 1 = f$. Associativity is an immediate consequence of compatibility; more precisely, using vanishing, the composite of three maps may be reduced to poly-composing the three maps and then doing a trace on the two intermediate U-variables simultaneously, by tracing on their tensor product. (We shall leave this as an easy exercise.)

Next we must define the linearly distributive structure. It will be simpler to do this *via* a short detour. Given an object U in U, we can define a functor $U : \mathbf{X} \to \mathbf{X}[\mathsf{U}]$ which takes an object A to the object $([U], A)$ (which we denote (U, A), dropping the sequence bracket in the case of a singleton sequence). For a map $f : A \to B$, $U(f)$ is defined as $U \otimes A \xrightarrow{m} U \oplus A \xrightarrow{1 \oplus f} U \oplus B$, or equivalently, $U \otimes A \xrightarrow{1 \otimes f} U \otimes B \xrightarrow{m} U \oplus B$. That we are using the same notation for the object U and the induced functor ought not cause confusion, context making the intended meaning clear.

Lemma 22. $U(\_)$ *as defined above is a functor.*

Proof. (of the lemma) Clearly U preserves identity maps, by definition. To see that it also preserves composition, consider $A \xrightarrow{f} B \xrightarrow{g} C$. Then $U(f);U(g)$ is the circuit on the left below:



and the circuit on the right is $U(f;g)$, so we are done. □

Clearly, this construction and lemma applies to any sequence $\vec{U}$ of objects of U. The case when $\vec{U} = [\ ]$ is the empty sequence is of particular importance: it gives us an (obviously full and faithful) embedding $J : \mathbf{X} \to \mathbf{X}[\mathsf{U}]$, which is our version of the embedding N of [20]. We remark here that in general the functor $\vec{U}$ is neither full nor faithful: for example, we shall see later that $(U, \top) \cong (U, \bot)$ (Lemma 23).

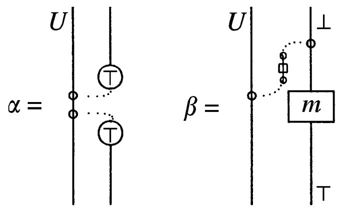
Now we return to the matter of the linearly distributive structure on $\mathbf{X}[\mathsf{U}]$. The tensor and par of the objects $([U_1, \ldots, U_n], A)$ and $([U_{n+1}, \ldots, U_{n+m}], B)$ are given by

$$([U_1, \ldots, U_n], A) \otimes ([U_{n+1}, \ldots, U_{n+m}], B) = ([U_1, \ldots, U_{n+m}], A \otimes B),$$
$$([U_1, \ldots, U_n], A) \oplus ([U_{n+1}, \ldots, U_{n+m}], B) = ([U_1, \ldots, U_{n+m}], A \oplus B).$$

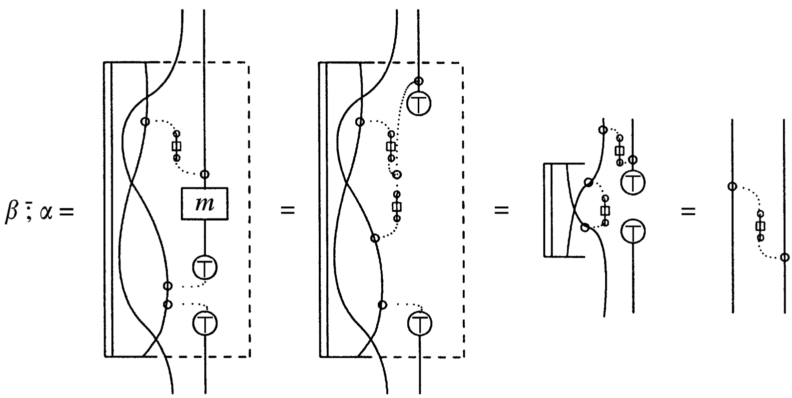
Note that by merely appending the lists of U objects in both cases, we are relying on the fact that these objects are all in the core. The units are given by the images under J of the corresponding units in $\mathbf{X}$: $\top = ([\ ], \top)$ and $\bot = ([\ ], \bot)$. The natural transformations for associativity, unit isomorphisms, and linear distributivities are all given by the images of the similarly named transformations in $\mathbf{X}$ under the appropriate functors $\vec{U}$, where $\vec{U}$ (the sequence, not the functor) is formed by concatenating the appropriate U sequences so that the domain and codomain work out right. An example will illustrate this: $\delta_L^L : (U, A) \otimes ((V, B) \oplus (W, C)) \to ((U, A) \otimes (V, B)) \oplus (W, C)$ is $[U, V, W](\delta_L^L)$. The point here is that since both tensor and par merely concatenate the U sequences, these natural transformations will have the same U sequences in domain and codomain, and so we can use this trick to extend them to $\mathbf{X}[\mathsf{U}]$. There is a minor complication with the symmetry maps: one must apply symmetry to the U component first. Consider $c_\otimes : (U, A) \otimes (V, B) \to (V, B) \otimes (U, A)$ for example. Since $(U, A) \cong (U, \top) \otimes ([\ ], A)$ (and in view of Lemma 23 below, $\cong (U, \bot) \otimes ([\ ], A) \cong (U, \top) \oplus ([\ ], A) \cong (U, \bot) \oplus ([\ ], A)$), we can identify the symmetry $c_\otimes$ with the tensor (or par) of the symmetries $c_0 : [U, V] \to [V, U]$ in the free monoid generated by the objects of U, and $c_1 : A \otimes B \to B \otimes A$ in $\mathbf{X}$. This means we can "decompose" any coherence diagram in $\mathbf{X}[\mathsf{U}]$ into one in the free monoid generated by U and one in $\mathbf{X}$. Since these will both commute, the diagram in $\mathbf{X}[\mathsf{U}]$ will too. Diagrams not involving the symmetries are even simpler, although the same trick will work: since the linearly distributive structure is given functorially, the required coherence diagrams are just the images of similar diagrams in $\mathbf{X}$ and so automatically commute. So $\mathbf{X}[\mathsf{U}]$ is indeed a linearly distributive category. So we have completed the proof of Theorem 21. □

Lemma 23. $(U,\top)\cong(U,\bot)$.

Proof. (of the lemma) The (inverse) maps are given as follows. $\alpha : U\otimes\top\to U\to U\oplus\bot$ represents $\alpha : (U,\top)\to(U,\bot)$, and $\beta : U\otimes\bot\xrightarrow{m}U\oplus\bot\xrightarrow{1\oplus m}U\oplus\top$ represents $\beta : (U,\bot)\to(U,\top)$. In circuits, these are as follows.



To see these are inverse, we first consider $\beta\,\bar{;}\,\alpha$. In circuits:

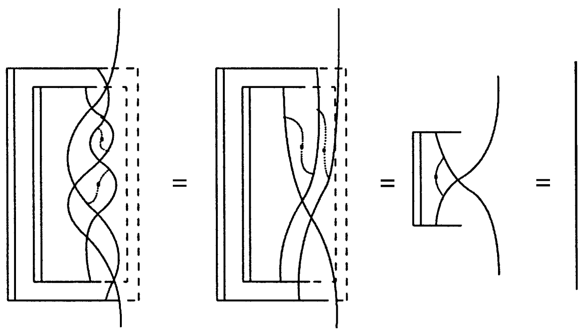


and the right-hand circuit is the identity. The reverse direction is similar, and will be left as an exercise. □

Proposition 24. *For each object U of* U, *$J(U)$ is complemented in* $\mathbf{X}[\mathsf{U}]$. *Moreover, under J, the trace on U becomes the complement trace on $J(U)$ in* $\mathbf{X}[\mathsf{U}]$.

Proof. The complement of U, or rather $([\ ],U)$, is given by $(U,\top)$, or equivalently, by the isomorphic $(U,\bot)$.

To show that $(U,\top)$ or $(U,\bot)$ is the complement of $([\ ],U)$, we just have to construct the appropriate τ and γ and show these satisfy the appropriate coherence conditions (Section 1.1.3). $\tau : ([\ ],\top) \to ([\ ],U) \oplus (U,\bot) \cong (U,U)$ is represented by the unit isomorphism $U \otimes \top \to U$. $\gamma : (U,\top) \otimes ([\ ],U) \cong (U,U) \to ([\ ],\bot)$ is represented by the unit isomorphism $U \to \bot \oplus U$. We have to show the composite $1 \otimes \tau ; \delta^L_L ; \gamma \oplus 1$ is (once the unit isomorphisms are "factored out") the identity on $(U,\top)$. (There is a similar dual condition giving the identity on $([\ ],U)$.) It is possible to eliminate the units from the calculation by grounding them; once we do that, this composite becomes the circuit on the left below, and we must reduce that to the identity on U. We use dotted arcs to represent MIX-barbells in order to save space.



And this concludes the proof of Proposition 24. □

Remark 25. It is a simple corollary that any (U,V) in $\mathbf{X}[\mathsf{U}]$, where U, V are in U, is complemented, with complement (V,U).

The construction of $\mathbf{X}[\mathsf{U}]$ is the universal solution to making a trace "canonical" (in the sense that complement traces are canonical). To state this precisely, we need some definitions. In the following, "preservation" is understood as being up to coherent isomorphisms.

Definition 26. Tr is the 2-category whose objects are pairs $\langle \mathbf{X},\mathsf{U}\rangle$, where $\mathbf{X}$ is a MIX category and U is a collection of pairwise compatibly traced objects of $\mathbf{X}$. A morphism $\langle \mathbf{X},\mathsf{U}\rangle \to \langle \mathbf{X}',\mathsf{U}'\rangle$ is a functor $F : \mathbf{X} \to \mathbf{X}'$ that preserves the linearly distributive category structure and for which $F(U)$ is an object of U' for each object U of U. Moreover, F must preserve the trace on each U. A 2-cell is a natural transformation that preserves the tensor and par, in the sense that $\alpha_{A\otimes B} = \alpha_A \otimes \alpha_B$, and similarly for par.

CTr is the full sub-2-category of Tr whose objects satisfy the property that the objects of U are all complemented, and whose traces are all complement traces. Again, the 2-cells are natural transformations preserving tensor and par.

It is straightforward to show that these are indeed 2-categories, and that in the case of CTr, if a functor preserves linearly distributive category structure, it must also preserve complements and so complement traces, so that the morphisms of CTr need only preserve linearly distributive category structure and send objects of U to U'. We shall denote the inclusion 2-functor i : $\mathsf{CTr} \to \mathsf{Tr}$, and the construction of $\mathbf{X}[\mathsf{U}]$ induces a 2-functor $\mathscr{I}$: $\mathsf{Tr} \to \mathsf{CTr}$. Then the following result is a direct analogue of the corresponding result in [20].

Proposition 27. *$\mathscr{I}$ is left biadjoint to i, and J induces the unit of this biadjunction.*

Proof. (*Sketch*) By "biadjoint" we mean (as usual) that the usual identities hold up to coherent isomorphism. In the following we shall suppress mention of the inclusion i when it is clear from the context.

We shall sketch the equivalence of the appropriate hom categories. Note that for $\langle \mathbf{X}, \mathsf{U}\rangle \in \mathsf{Tr}$, $\mathscr{I}\langle \mathbf{X}, \mathsf{U}\rangle = \langle \mathbf{X}[\mathsf{U}], J(\mathsf{U})\rangle$. Suppose given F : $\mathscr{I}\langle \mathbf{X}, \mathsf{U}\rangle \to \langle \mathbf{Y}, \mathsf{V}\rangle$ in CTr; define F^* : $\langle \mathbf{X}, \mathsf{U}\rangle \to \langle \mathbf{Y}, \mathsf{V}\rangle$ in Tr by $F^* = J;F$. For the reverse association, suppose given G : $\langle \mathbf{X}, \mathsf{U}\rangle \to \langle \mathbf{Y}, \mathsf{V}\rangle$ in Tr; then define G_* : $\mathscr{I}\langle \mathbf{X}, \mathsf{U}\rangle \to \langle \mathbf{Y}, \mathsf{V}\rangle$ in CTr by $G_*(([U_1, \ldots, U_n], A)) = G(U_1)^\perp \otimes \cdots \otimes G(U_n)^\perp \otimes G(A)$, where we denote the complement of an object V by $V^\perp$. On morphisms G_* is easily induced by the canonical construction of a morphism $W^\perp \otimes B \to V^\perp \otimes C$ from a morphism $V \otimes B \to W \otimes C$, for objects V, W in the core. (This requires the preservation of linearly distributive structure by G.) It is a straightforward matter to verify that F^* and G_* satisfy the required conditions to be 1-cells in the appropriate 2-category, and that these associations are binatural.

Next we must verify that $(F^*)_* \cong F$ and $(G_*)^* \cong G$. To check the former, we note that since $([U_1, \ldots, U_n], A) \cong (U_1, \top) \otimes \cdots \otimes (U_n, \top) \otimes ([\ ], A)$, it suffices to verify that $(F^*)_*((U, \top)) \cong F((U, \top))$ and that $(F^*)_*(([\ ], A)) \cong F(([\ ], A))$. Since F preserves linearly distributive structure, the first isomorphism essentially reduces to the fact that the complement of $([\ ], U)$ is $(U, \top)$. The second isomorphism is trivial, and essentially reduces to the fact that the complement of $\perp$ is $\top$. This remark also suffices to show that $(G_*)^*(A) \cong G(A)$. This completes the sketch of the proof. $\square$

5. Fixpoint combinators

There is a well-understood connection between trace operators and fixpoint operators in the context of cartesian categories (see Hasegawa [15] for example). In this section we investigate this connection more locally and in a more general setting, namely by considering having a fixpoint operator on a given object in a MIX category. There will be some additional structure we must impose, as we shall see below, and further, there are some subtle variations on the more familiar context. Note that we refer to "fixpoint operators", but our definition does not postulate the usual fixpoint equations. Indeed, without some modest further conditions, these need not be satisfied, but we shall see that the present notion does capture the essence of fixpoint operators.

Definition 28. Suppose $\mathbf{X}$ is a MIX category, U an object of $\mathbf{X}$. We say U has a *fixpoint combinator* if there is a morphism $e : U \to \perp$ in $\mathbf{X}$ and for each pair of objects A, B of $\mathbf{X}$ a map fix_U^{AB} : $\mathrm{Hom}(U \otimes A, U \oplus B) \to \mathrm{Hom}(A, U \oplus B)$ making *fix* a strong combinator. These must satisfy the following conditions:

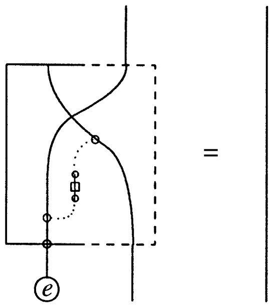
$$\text{Yanking}: \text{fix}(c_{\otimes}; m_{UU}); e \otimes 1; u^L_{\oplus} = 1_U$$

$$\text{Fixed compatibility}: \text{fix}(\text{fix}(f)) = \text{fix}(\text{fix}(c_{\otimes} \otimes 1; f))$$

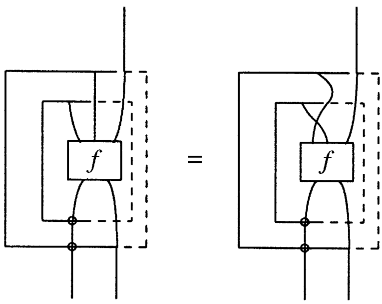
The first axiom, yanking, is just the yanking axiom for trace operators: under the association between traces and fixpoint operators, these yanking axioms correspond to each other. In the fixed compatibility axiom, f is a morphism $U \otimes U \otimes A \to U \oplus B$, and we require the iterated fix of this to be equal to the similarly iterated fix of the morphism $U \otimes U \otimes A \xrightarrow{c\otimes 1} U \otimes U \otimes A \xrightarrow{f} U \oplus B$. We have suppressed some instances of associativity here. This fixed compatibility axiom is a variant of compatibility for traces. The point of this axiom is that the "fixed" output is the same for both instances of the fixpoint combinator (this ought to be clearer in the circuit diagram below). Later we shall give a definition of (ordinary) compatibility for fixpoint combinators in which each fixpoint combinator will have a different "fixed" output; this will correspond to the compatibility condition as given before for trace operators.

Note that by requiring fix to be a strong combinator, we are requiring that it satisfies the evident versions of tightening and superposing. We shall list the axioms below in circuit form. We use a box notation similar to that for trace operators for the fixpoint combinator, but indicate the output for the U that has been "fixed" with a small circle. (This may be regarded as the "principal port" of the fixpoint box.) Also, we denote e followed by the terminal node for $\perp$ by a terminal e node (i.e. a node without output wires), as we did for γ earlier in the paper.

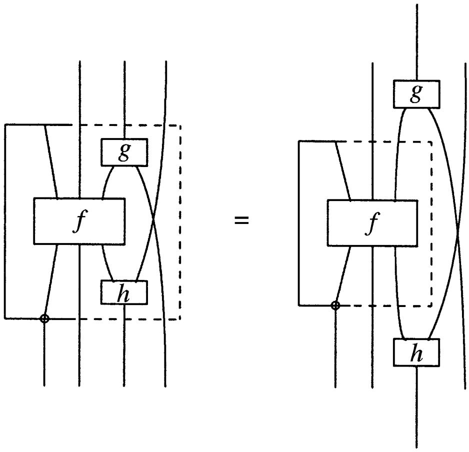
Yanking:



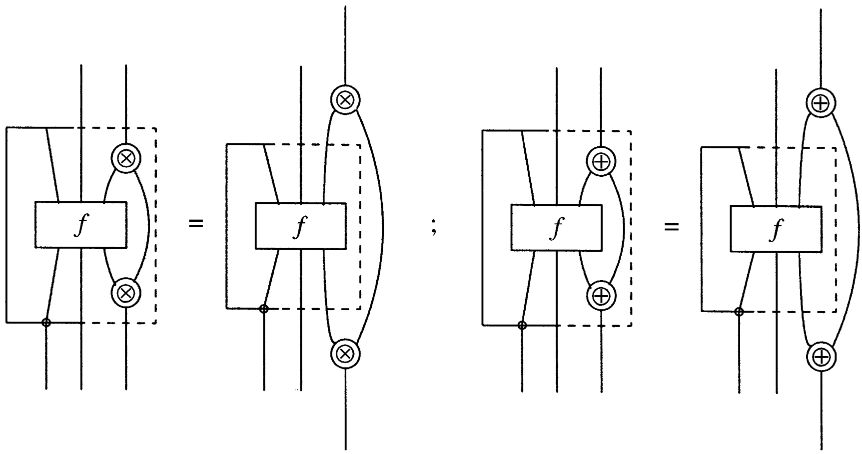
Fixed compatibility:



Tightening:

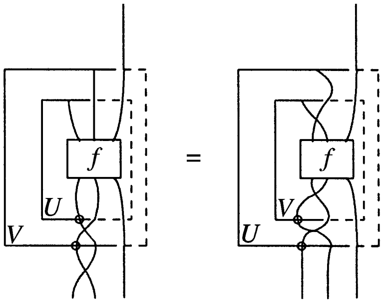


Superposing:



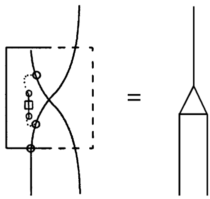
Of course, there is an alternate version of superposing, corresponding to the alternate version for traces.

Definition 29. Suppose $\mathbf{X}$ is a MIX category, and U, V objects of $\mathbf{X}$ each with a fixpoint combinator, say fix_U, fix_V. These are called *compatible* if for any $f : U \otimes V \otimes A \to U \oplus V \oplus B$, $\mathrm{fix}_V(\mathrm{fix}_U(f); c_\oplus \otimes 1); c_\oplus \otimes 1 = \mathrm{fix}_U(\mathrm{fix}_V(f'); c_\oplus \otimes 1)$, where $f' = V \otimes U \otimes A \stackrel{c\otimes 1}{\to} U \otimes V \otimes A \stackrel{f}{\to} U \oplus V \oplus B \stackrel{c\oplus 1}{\to} V \oplus U \oplus B$. (We have suppressed some evident uses of associativity here.) In circuits, this is the following equation:



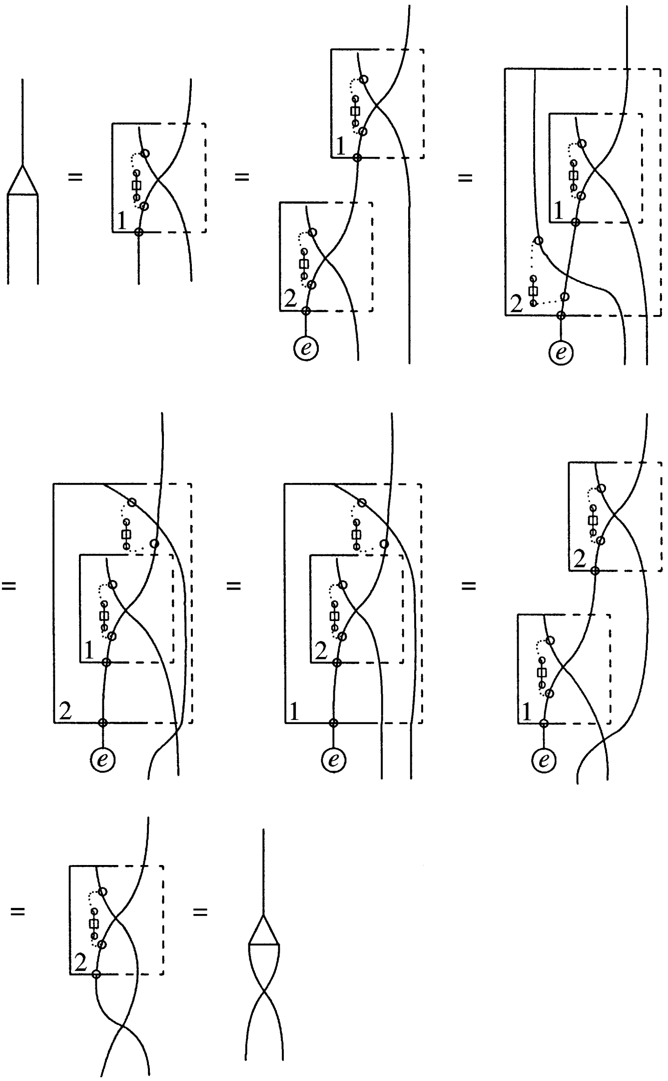
Remark 30. We shall see in Corollary 34 below that from the compatibility condition given above, it follows that there is another compatibility condition that must hold between compatible fixpoint combinators fix_1, fix_2 defined on the same object U, viz. the two-combinator version of fixed compatibility $\mathrm{fix}_1(\mathrm{fix}_2(f)) = \mathrm{fix}_2(\mathrm{fix}_1(c_\otimes \otimes 1; f))$, for a map $f : U \otimes U \otimes A \to U \oplus B$.

Now we note that given a fixpoint combinator on U, there is an induced cocommutative comonoid (with respect to $\oplus$) structure on U. The comultiplication is given by the morphism Δ defined by $\mathrm{fix}(m_{UU}; c_\oplus) : U \to U \oplus U$. Note that this is equal to $\mathrm{fix}(c_\otimes; m_{UU})$. (We shall show later that if U has a fixpoint combinator, it also has a trace, and so is in the core. So if we wanted, we could define this comultiplication, and so the comonoid structure, with respect to the tensor $\otimes$.) We shall occasionally denote this as follows.



Lemma 31. *Δ is cocommutative.*

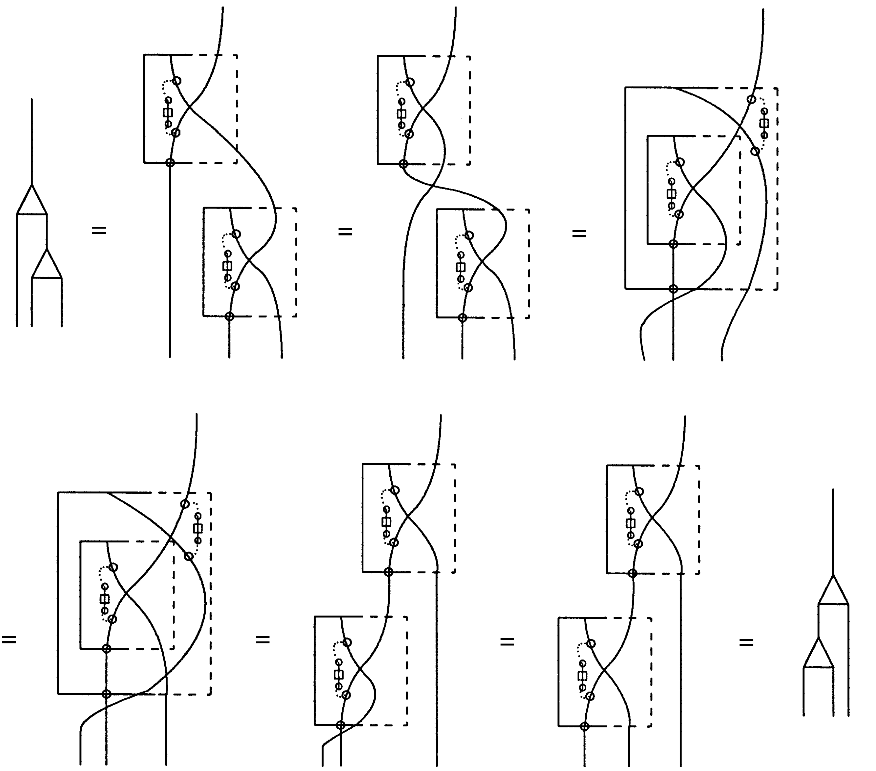
Proof. This is most simply shown by the following circuit rewrites (for the moment, ignore the small indices — their role will appear below in the proof of Corollary 34).



The key step in the proof above is in the middle of the second line, where use is made of fixed compatibility. In addition, some rewiring and some playing around with the "twist" maps $c_\otimes$, $c_\oplus$ has been done silently, most importantly, moving $c_\oplus$ inside the boxes and then past the barbell to cancel a twist introduced by the fixed compatibility. Moving a twist past a barbell uses the identity $c\,;m = m\,;c$ which we have seen before. $\square$

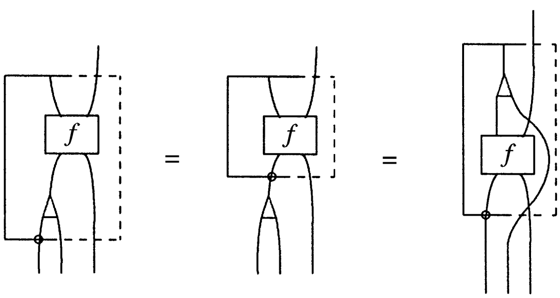
Proposition 32. (U, e, Δ) *is a cocommutative comonoid* (*with respect to* $\oplus$).

Proof. We have seen that Δ is cocommutative; using this it is easy to show e is a unit for Δ, since yanking gives one side, and cocommutativity then allows us to derive the other side. So it only remains to show that Δ is coassociative:

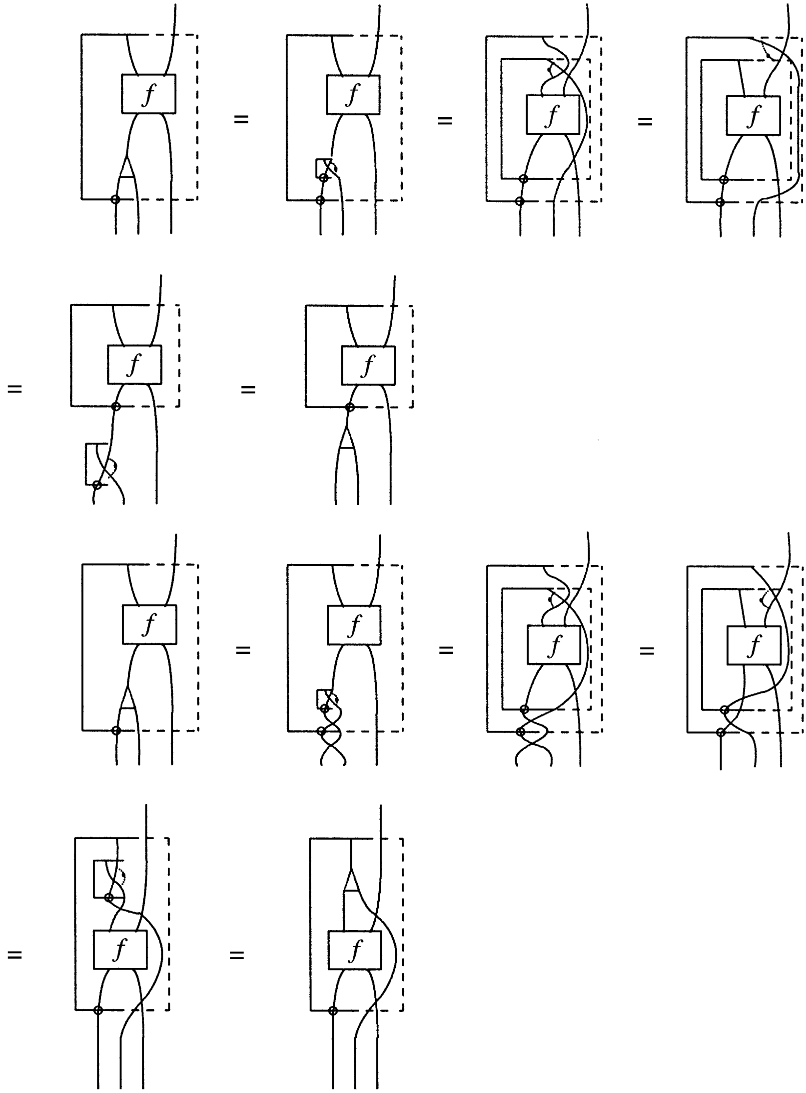


□

Lemma 33. *Given any self-compatible fixpoint combinator on an object U, we have the following two equations*:



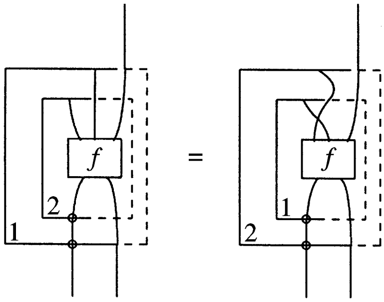
Proof. The following circuit rewrites show these equations; note that we have again represented the MIX-barbell with a dotted arc to save space.



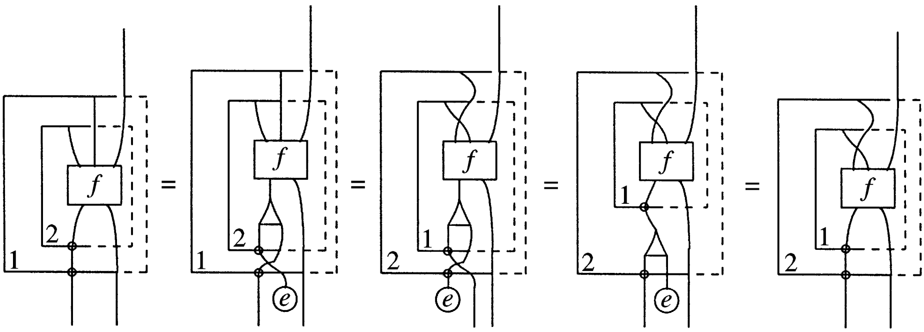
□

There are several corollaries that we can derive from these equations.

Corollary 34. *Given two compatible fixpoint combinators* fix_1 *and* fix_2 *on the same object* U, *the following variant of fixed compatibility holds*: $\mathrm{fix}_1(\mathrm{fix}_2(f)) = \mathrm{fix}_2(\mathrm{fix}_1(c_\otimes \oplus 1; f))$ *for a map* $f : U \otimes U \otimes A \to U \oplus B$. *In circuits, this is the following.*



Proof. In the following we use the Δ notation as defined above. As we shall use Lemma 33 to "pull" this Δ outside a fixpoint box for fix_2, it will be necessary to use the Δ for fix_2, though it is simple to show that Δ is independent of such a choice. (This fact will also follow from the next corollary.)



□

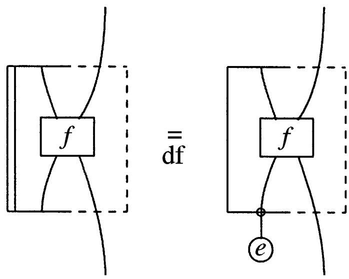
Corollary 35. *Any two compatible fixpoint combinators on U are equal.*

Proof. Notice the small indices in the proof that Δ is cocommutative (Lemma 31). If these identify two fixpoint combinators, then the step in the second line involving fixed compatibility is valid if the combinators are compatible, according to Corollary 34. Then we see that the two combinators must be equal. Note that this result will also follow from the corresponding result for compatible trace operators, once we have established the correspondence between fixpoint combinators and trace operators. □

Theorem 36. *Suppose* **X** *is a* MIX *category, and U an object of* **X**. *The following are equivalent.*

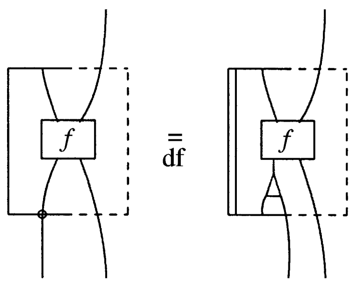
(i) *U has a self-compatible trace operator and a cocommutative comonoid structure* (*with respect to* $\oplus$).
(ii) *U has a self-compatible fixpoint combinator.*

Proof. Given a fixpoint combinator fix on U, we define a trace operator tr^{f} by $\mathrm{tr}^{\mathrm{f}}(f) = \mathrm{fix}(f) ⨟ e$. Using circuits, this is



Clearly, yanking for this trace operator is given by yanking for the fixpoint combinator, and tightening, superposing, and self-compatibility are similarly induced by the same properties of the fixpoint combinator. U has a cocommutative comonoid structure (Proposition 32).

For the reverse direction, if we have a self-compatible trace operator tr on U with the stated properties, then we can define a fixpoint combinator by $\mathrm{fix}^{\mathrm{t}}(f) = \mathrm{tr}(f ⨟ \Delta)$. In circuits:



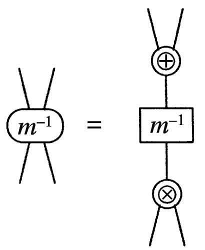
Again, yanking, tightening, and superposing are easy consequences of the corresponding equations for the trace operator. Self-compatibility likewise is straightforward, but to show fixed compatibility we need to use self-compatibility of the trace operator plus cocommutativity and coassociativity of Δ to get the wires arranged in the correct manner. This is a simple exercise similar to the many such calculations we have already seen.

Finally, we want to show that these constructions are inverse. One direction is trivial: starting from a trace operator, the trace operator induced by the induced fixpoint combinator is clearly the original trace operator, since e is a unit for Δ: $\mathrm{tr}^{\mathrm{ft}}(f) =$

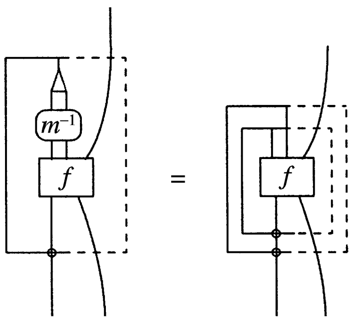
$\mathrm{fix}^{\mathsf{t}}(f) ⨾ e = \mathrm{tr}(f ⨾ \Delta) ⨾ e = \mathrm{tr}(f) ⨾ \Delta ⨾ e = \mathrm{tr}(f)$. For the reverse direction, we need an application of Lemma 33 to move the Δ (which is attached to the principal port of the fixpoint box) outside the fixpoint box so that it may be cancelled: $\mathrm{fix}^{\mathsf{tf}} = \mathrm{tr}^{\mathsf{f}}(f ⨾ \Delta) = \mathrm{fix}(f ⨾ \Delta) ⨾ e = \mathrm{fix}(f) ⨾ \Delta ⨾ e = \mathrm{fix}(f)$. $\square$

Before leaving this section, we point out some properties that are characteristic of fixpoint combinators (and leave some others as exercises for the reader).

To begin with, we need some notation. If we want to consider Δ as defined on the $\otimes$ structure, we must put an instance of m^{-1} following Δ, so as to change the implicit $\oplus$ into an $\otimes$, just as we did with the τ when we defined the complement trace (Proposition 9). To save space, we shall denote this use of m^{-1} by an oval-shaped box — this is not the usual sort of component box, since its input wires are par'ed and its output wires are tensored. So this means that the oval on the left below is an abbreviation for the graph on the right.

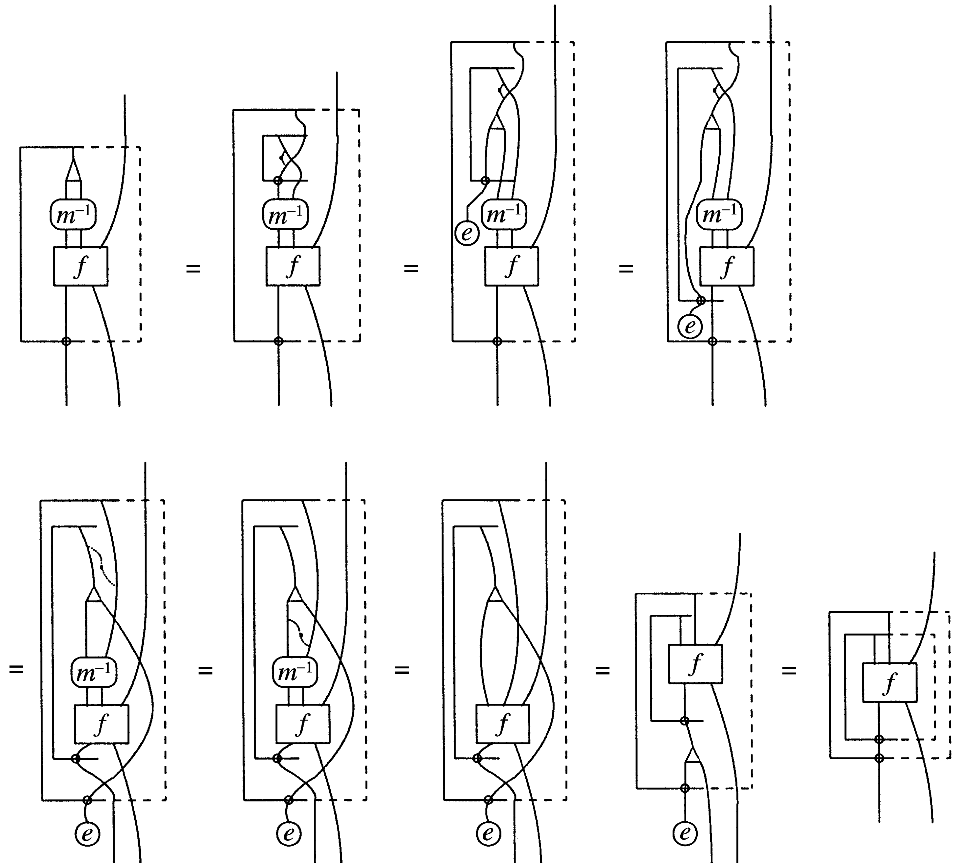


Lemma 37 (The diagonal property). *Suppose* fix *is a self-compatible fixpoint combinator on* U, $f : U \otimes U \otimes A \to U \oplus B$. *Then* $\mathrm{fix}(\Delta ⨾ f) = \mathrm{fix}(\mathrm{fix}(f))$. *In circuits*:



We could make this statement somewhat more elegant by defining a new Δ operator that contained both the "old" Δ and the m^{-1} oval. As we have no further need of this "new" Δ we see no real need for this definition, however.

Proof. In the circuits below, we again represent the MIX-barbell by a dotted wire.



□

We end this section with a derivation of the fixpoint property, which we ought to expect a fixpoint combinator to satisfy. It turns out, however, that for this we need a further property of Δ, namely that it be "natural" in the following sense. (Recall that since we are presenting this "locally", for a fixed U, Δ is not a natural transformation; the property we now want would be a consequence of Δ being a natural transformation.) For simplicity, we begin with the "categorical" (one input, one output) case.

Definition 38. Suppose U has a fixpoint combinator; let Δ be the induced comultiplication map. Δ is said to be *natural* if for any $f : U \to U$, $\Delta; f \oplus f = f; \Delta : U \to U \oplus U$.

Proposition 39. *Suppose U has a fixpoint combinator* fix *for which the induced comultiplication Δ is natural. Then for any $f : U \to U$,* $\mathrm{fix}(f); f = \mathrm{fix}(f) : \top \to U$.

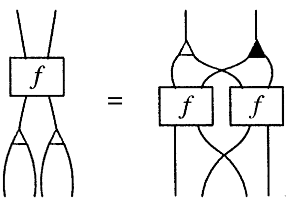
Proof. It is simple to construct a circuit proof of this — it may be considered a special case of the next proposition in any event. The following equations ought to provide the necessary hint. $\mathrm{fix}(f); f = \mathrm{fix}(f;\Delta); e \oplus f = \mathrm{fix}(\Delta; f \oplus f) \fatsemi e = \mathrm{fix}(f;\Delta) \fatsemi e = \mathrm{fix}(f)$. □

For the general ("polycategorical") case, we need to generalise the notion of "naturality" for Δ. First, we shall try to simplify the notation. As we did in our discussion of $\mathbf{X}[\mathsf{U}]$, and without loss of generality, we shall avoid notational clutter by using the "two input – two output" case as generic. So we shall consider a map $f : U \otimes A \to U \oplus B$ as an arbitrary component (or polymorphism). We shall state the next definition in this "two input – two output" style, but with obvious modifications, this may be restated in full generality, which is our intended meaning.

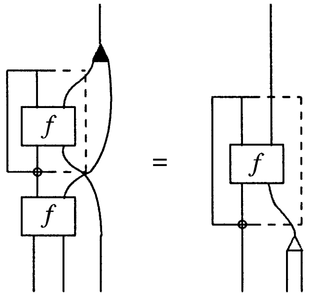
Definition 40. Suppose U has a fixpoint combinator; let Δ be the induced comultiplication map. Suppose f is an arbitrary morphism as above; we shall denote it as $f : U \otimes A \to U \oplus B$. Suppose A (i.e. each input other than the initial U) has a "$\otimes$-duplication" map $\blacktriangle : A \to A \otimes A$, and B (i.e. each output other than the initial U) has a "$\otimes$-duplication" map $\Delta : B \to B \otimes B$. Δ is said to be *polynatural* for f if the following diagram commutes:

$$\begin{tikzcd}
U \otimes A \arrow[r, "f"] \arrow[d, "\Delta \otimes \blacktriangle"'] & U \otimes B \arrow[d, "\Delta \otimes \Delta"] \\
(U \oplus U) \otimes (A \otimes A) \arrow[d, "\delta'"'] & U \oplus U \oplus B \oplus B \\
(U \otimes A) \oplus (U \otimes A) \arrow[r, "f \oplus f"] & U \oplus B \oplus U \oplus B \arrow[u, "1 \oplus c \oplus 1"']
\end{tikzcd}$$

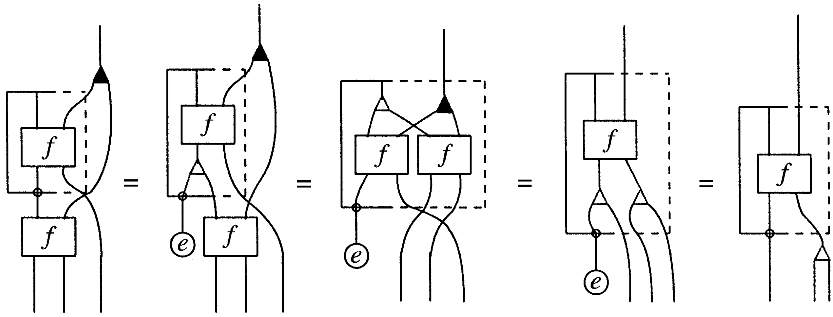
where δ' is the evident linear distributivity (plus some symmetry and associativity). In circuits this is the following. We represent $\blacktriangle$ as a link — this functions like a tensor link (and in particular has the same switches as a tensor link).



Proposition 41. *Suppose U has a fixpoint combinator, Δ the induced comultiplication, and $f : U \otimes A \to U \oplus B$ is a morphism for which Δ is polynatural.* (*This includes some structural assumptions on A and B, as in Definition* 40.) *Then* $\blacktriangle ; \mathrm{fix}(f) ; f = \mathrm{fix}(f) ; \Delta : A \to U \otimes B \oplus B$. *In circuits, this is the following*:



Proof.



□

Remark 42. There is a question of what is the most appropriate level of generality for the fixpoint property. We have stated it in a minimalist form: the conditions necessary are assumed, and no more. However, the most suitable context for this result would seem to be something a bit stronger; perhaps a cartesian linearly distributive category. If we suppose also that the product is both tensor and par, then we are essentially in the context of [15].

Remark 43. There is another property of fixpoint operators that one often encounters, namely the Bekič property. It is well-known that in the usual cartesian context, this property is true of any fixpoint operator that is induced by a trace operator (see [15] for instance). In the present context, it may be shown that any fixpoint combinator satisfies this property. (It was in fact by establishing the Bekič property for traces that we were led to the correspondence of Theorem 36.) However, as we have no need of this result here, and it involves some lengthy circuit calculations, we are happy to leave it as a pleasant exercise for the reader.

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