Weakly distributive categories

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Abstract

There are many situations in logic, theoretical computer science, and category theory where two binary operations—one thought of as a (tensor) "product", the other a "sum"—play a key role, such as in distributive categories and in $*$-autonomous categories. (One can regard these as essentially the AND/OR of traditional logic and the TIMES/PAR of (multiplicative) linear logic, respectively.) In the latter example, however, the distributivity one often finds is conspicuously absent: in this paper we study a "linearisation" of distributivity that *is* present in this context. We show that this weak distributivity is precisely what is needed to model Gentzen's cut rule (in the absence of other structural rules), and show how it can be strengthened in two natural ways, one to generate full distributivity, and the other to generate $*$-autonomous categories.

0. Introduction

There are many situations in logic, theoretical computer science, and category theory where two binary operations, "tensor products" (though one may be a "sum"), play a key role. The multiplicative fragment of linear logic is a particularly interesting example as it is a Gentzen style sequent calculus in which the structural rules of contraction, thinning, and (sometimes) exchange are dropped. The fact that these rules are omitted considerably simplifies the derivation of the cut elimination theorem. Furthermore, the proof theory of this fragment is interesting and known [Se89] to correspond to $*$-autonomous categories as introduced by Barr in [Ba79].

In the study of categories with two tensor products one usually assumes a distributivity condition, particularly in the case when one of these is either the product or sum. The multiplicative fragment of linear logic (*viz.* $*$-autonomous categories) is a significant exception to this situation; here the two tensors "times" ($\otimes$) and "par" ($⅋$, in this paper denoted $\oplus$—note that this conflicts with Girard's notation) do not distribute one over the other.

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However, $*$-autonomous categories are known to satisfy a weak notion of distributivity. This weak distributivity is given by maps of the form:

$$A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

$$A \otimes (B \oplus C) \longrightarrow B \oplus (A \otimes C)$$

(and two other versions should the tensors lack symmetry.)

These maps, interpreted as entailments, are also valid in what might be considered the minimal logic of two such tensors, namely the classical Gentzen sequent calculus with the left and right introduction rules for conjunction and disjunction and with cut as the only structure rule. This Gentzen style proof theory has a categorical presentation already in the literature, *viz.* the polycategories of Lambek and Szabo [Sz75]. It should therefore be possible to link $*$-autonomous categories and polycategories. However, this begs a wider question of precisely what properties a category must satisfy to be linked in this manner to the logical superstructure provided by a polycategory.

It turns out that these weak distributivity maps, when present coherently, are precisely the necessary structure required to construct a polycategory superstructure, and whence a Gentzen style calculus, over a category with two tensors. The weak distributivity maps allow the expression of the Gentzen cut rule in terms of ordinary (categorical) composition.

We call categories with two tensors linked by coherent weak distribution *weakly distributive categories*. They can be built up to be the proof theory of the full multiplicative fragment of classical linear logic[1] by coherently adding maps

$$\top \longrightarrow A \oplus A^{\perp}$$

$$A \otimes A^{\perp} \longrightarrow \perp$$

(and symmetric duals as necessary), or to the proof theory of the $\wedge, \vee$ fragment of intuitionistic propositional logic by coherently adding contraction, thinning, and exchange. The former corresponds to $*$-autonomous categories and the latter to distributive categories. In fact, weakly distributive categories lie at the base of a rich logical hierarchy, unifying several hitherto separate developments in the logics of theoretical computer science.

One point must be made about the connection with linear logic. A novel feature of our presentation is that we have considered the two tensor structure separately from the structure given by linear negation $(-)^{\perp}$. We show how to obtain the logic of $*$-autonomous categories from that of weakly distributive categories, giving, in effect, another presentation of $*$-autonomous categories. It sometimes happens

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[1] The system *FILL* (full intuitionistic linear logic) of dePaiva [dP89] amounts to having just the second of these (families of) maps. From the autonomous category viewpoint, these are the more natural maps, as they correspond to evaluations. The symmetry of the $*$-autonomous viewpoint then suggests the first (family of) maps.

that it is easier to verify $*$-autonomy this way; for example, verifying that a lattice with appropriate structure is $*$-autonomous becomes almost trivial if one checks the weak distributivity first (see [Ba91].)

In this short version of our paper, we shall not have space to say as much as we would like about models of this structure. However, one should note that $*$-autonomous categories, distributive categories, braided monoidal categories, among others, are all weakly distributive. In addition, the opposite of a weakly distributive category is weakly distributive (with the tensors changing roles), so, for example, co-distributive categories are weakly distributive. One has frequently been struck by the strangeness of the distributivity in such co-distributive categories as the category of commutative rings, or the category of distributive lattices, and so on— they may now be seen as weakly distributive in the standard manner.

We have also been very brief about coherence questions here; further, we plan to show how to associate a term calculus to these categories. (This term calculus is based on a calculus developed in [Co89] for symmetric monoidal categories.) We plan to elaborate on all these matters elsewhere. (Added in proof: Coherence has been treated in the recent work of Blute and Seely [BS91], as well as a partial answer to the question of the conservativity of the extension to $*$-autonomous categories. In that paper, coherence is completely settled, but conservativity is only shown for the fragment without units.)

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1. Polycategories

We shall begin with a review of Szabo's notion of a polycategory:

Definition 1.1 *A polycategory* **C** *consists of a set* $Ob(\mathbf{C})$ *of objects and a set* $M\varphi(\mathbf{C})$ *of morphisms, (also called arrows, polymorphisms, ...) just like a category, except that the source and target of a morphism are finite sequences of objects*

$$source\colon M\varphi(\mathbf{C}) \longrightarrow Ob(\mathbf{C})^*$$

$$target\colon M\varphi(\mathbf{C}) \longrightarrow Ob(\mathbf{C})^*$$

where X^* = *the free monoid generated by* X.

There are identity morphisms $i_A\colon A \to A$ *between singleton sequences only and a notion of composition given by the* **cut rule***:*

$$\frac{\Gamma_1, A, \Gamma_2 \xrightarrow{g} \Gamma_3 \quad \Delta_1 \xrightarrow{f} \Delta_2, A, \Delta_3}{\Gamma_1, \Delta_1, \Gamma_2 \xrightarrow{g\ {}_i{\circ}_j\ f} \Delta_2, \Gamma_3, \Delta_3}$$

where the length of Γ_1 is i and the length of Δ_2 is j. When the subscripts are clear from the context they shall be dropped.

We have the following equations:

1. $$\Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3 \quad = \quad \dfrac{A \xrightarrow{i_A} A \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Gamma_1 \xrightarrow{i_A\ {}_0\circ_j\ f} \Gamma_2, A, \Gamma_3}$$

2. $$\Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \quad = \quad \dfrac{\Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \quad A \xrightarrow{i_A} A}{\Gamma_1, A, \Gamma_2 \xrightarrow{f\ {}_i\circ_0\ i_A} \Gamma_3}$$

3. $$\dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \qquad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3, B, \Delta_4 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Delta_1, \Gamma_1, \Delta_2 \xrightarrow{g\ {}_i\circ_j\ f} \Gamma_2, \Delta_3, B, \Delta_4, \Gamma_3}}{\Phi_1, \Delta_1, \Gamma_1, \Delta_2, \Phi_2 \xrightarrow{h\ {}_l\circ_{j+m}\ (g\ {}_i\circ_j\ f)} \Gamma_2, \Delta_3, \Phi_3, \Delta_4, \Gamma_3}$$

$$= \quad \dfrac{\dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3, B, \Delta_4}{\Phi_1, \Delta_1, A, \Delta_2, \Phi_2 \xrightarrow{h\ {}_l\circ_m\ g} \Delta_3, \Phi_3, \Delta_4} \qquad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Delta_1, \Gamma_1, \Delta_2, \Phi_2 \xrightarrow{(h\ {}_l\circ_m\ g)\ {}_{l+i}\circ_j\ f} \Gamma_2, \Delta_3, \Phi_3, \Delta_4, \Gamma_3}$$

4. $$\dfrac{\dfrac{\Phi_1, A, \Phi_2, B, \Phi_3 \xrightarrow{h} \Phi_4 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Gamma_1, \Phi_2, B, \Phi_3 \xrightarrow{h\ {}_i\circ_j\ f} \Gamma_2, \Phi_4, \Gamma_3} \qquad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Phi_1, \Gamma_1, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{(h\ {}_i\circ_j\ f)\ {}_{i+k+l}\circ_m\ g} \Delta_2|\Gamma_2, \Phi_4, \Delta_3|\Gamma_3}$$

$$= \quad \dfrac{\dfrac{\Phi_1, A, \Phi_2, B, \Phi_3 \xrightarrow{h} \Phi_4 \quad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Phi_1, A, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{h\ {}_{i+1+l}\circ_m\ g} C} \qquad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Gamma_1, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{(h\ {}_{i+1+l}\circ_m\ g)\ {}_i\circ_j\ f} \Delta_2|\Gamma_2, \Phi_4, \Delta_3|\Gamma_3}$$

provided at least one of Γ_2, Δ_2 is empty, and at least one of Γ_3, Δ_3 is empty, so $\Delta_i|\Gamma_i$ represents the trivial concatenation of a sequence and an empty sequence.

5. $$\dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \qquad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3, B, \Gamma_4}{\Delta_1, \Gamma_1, \Delta_2 \xrightarrow{g\ {}_i\circ_j\ f} \Gamma_2, \Delta_3, \Gamma_3, B, \Gamma_4}}{\Phi_1|\Delta_1, \Gamma_1, \Phi_2|\Delta_2 \xrightarrow{h\ {}_m\circ_{j+k+l}\ (g\ {}_i\circ_j\ f)} \Gamma_2, \Delta_3, \Gamma_3, \Phi_3, \Gamma_4}$$

$$= \quad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3 \qquad \dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3, B, \Gamma_4}{\Phi_1, \Gamma_1, \Phi_2 \xrightarrow{h\ {}_m\circ_{j+1+l}\ f} \Gamma_2, A, \Gamma_3, \Phi_3, \Gamma_4}}{\Phi_1|\Delta_1, \Gamma_1, \Phi_2|\Delta_2 \xrightarrow{g\ {}_i\circ_j\ (h\ {}_m\circ_{j+1+l}\ f)} \Gamma_2, \Delta_3, \Gamma_3, \Phi_3, \Gamma_4}$$

where again at least one of Φ_1, Δ_1 *is empty, and at least one of* Φ_2, Δ_2 *is empty.*

Remark 1.2 (Planar Polycategories)

There is a certain amount of permutation built into the cut rule and this results in the restrictions we have had to place on some of the equations. Lambek [La90] has given several weaker variants in which the restrictions are built directly into the cut rule so that this permutation is avoided: a similar system was presented by G.L. Mascari at the Durham Symposium. In the weakest system, **BL1**, cut is restricted to instances where either $\Gamma_1 = \Gamma_2 = \phi$ or $\Delta_2 = \Delta_3 = \phi$; this corresponds to having no weak distributivities, in the sense of the next section. A stronger system, **BL2**, in addition also allows cuts where either $\Gamma_1 = \Delta_3 = \phi$ or $\Gamma_2 = \Delta_2 = \phi$, corresponding in our setting to having only the "non permuting" distributivities δ_L^L and δ_R^R. We shall call the notion of polycategory based on **BL2**, allowing only cuts where either Γ_1 or $\Delta_2 = \phi$, and either Γ_2 or $\Delta_3 = \phi$, a "planar polycategory" (though it is not necessarily a polycategory!), as it corresponds to planar non commutative linear logic. (Note that in a planar polycategory, the restrictions on the equations are unnecessary.)

By allowing unrestricted cut in our setup, we are in effect introducing two "permuting" weak distributivities, δ_R^L and δ_L^R, as defined in the next section. So, in a sense, we are not dealing with a strictly non commutative logic. For example, if the weak distribution rules are inverted, the permuting ones will give a braiding structure in general. Thus we are generalizing braided monoidal categories (rather than symmetric or general non symmetric monoidal categories). We shall return to this point elsewhere. □

Next, we define a polycategory with two tensors: this amounts to having two binary operations $\otimes, \oplus$ on objects, extended to morphisms according to the following inference rules:

$$(\otimes\ \mathrm{L})\ \frac{\Gamma_1, A, B, \Gamma_2 \xrightarrow{f} \Gamma_3}{\Gamma_1, A \otimes B, \Gamma_2 \xrightarrow{f^{\otimes_i}} \Gamma_3} \qquad (\otimes\ \mathrm{R})\ \frac{\Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3 \qquad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Gamma_1, \Delta_1 \xrightarrow{f\ {}_i\otimes_j\ g} \Gamma_2|\Delta_2, A \otimes B, \Gamma_3|\Delta_3}$$

provided (in ($\otimes$ R)) at least one of Γ_2, Δ_2 is empty and at least one of Γ_3, Δ_3 is empty. In ($\otimes$ L), $i =$ length of Γ_1; in ($\otimes$ R), $i =$ length of Γ_2, $j =$ length of Δ_2, (so $ij = 0$).

$$(\oplus\ \mathrm{L})\ \frac{\Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \qquad \Delta_1, B, \Delta_2 \xrightarrow{g} \Delta_3}{\Gamma_1|\Delta_1, A \oplus B, \Gamma_2|\Delta_2 \xrightarrow{f\ {}_i\oplus_j\ g} \Gamma_3, \Delta_3} \qquad (\oplus\ \mathrm{R})\ \frac{\Gamma_1 \xrightarrow{f} \Gamma_2, A, B, \Gamma_3}{\Gamma_1 \xrightarrow{f^{\oplus_i}} \Gamma_2, A \oplus B, \Gamma_3}$$

provided (in ($\oplus$ L)) at least one of Γ_1, Δ_1 is empty and at least one of Γ_2, Δ_2 is empty. In ($\oplus$ L), i = length of Γ_1, and j = length of Δ_1, (so $ij = 0$); in ($\oplus$ R), i = length of Γ_2.

(Note that we have indexed the labels as we did with cut; when clear from the context, we shall drop these subscripts.)

There are many further equivalences of derivations as in Definition 1.1. These can be considerably simplified if we give the following equivalent formulation of the tensor rules:

Definition 1.3 *A two-tensor-polycategory is a polycategory with two binary operations* $\otimes, \oplus$ *on objects, with morphisms*

$$m_{AB} : A, B \longrightarrow A \otimes B$$

$$w_{AB} : A \oplus B \longrightarrow A, B$$

and the rules of inference ($\otimes$ L) and ($\oplus$ R) above. These rules are to represent bijections stable under cut, so the following equations must hold:

- $m^{\otimes} = i$
- $g \circ f^{\otimes} = (g \circ f)^{\otimes}$ *for* $g : \Delta_1, C, \Delta_2 \longrightarrow \Delta_3$ *and* $f : \Gamma_1 \longrightarrow \Gamma_2, C, \Gamma_3$, *and where* Γ_1 *contains the sequence* A, B.
- $f^{\otimes} \circ g = (f \circ g)^{\otimes}$ *for* $g : \Delta_1 \longrightarrow \Delta_2, C, \Delta_3$ *and* $f : \Gamma_1, C, \Gamma_2 \longrightarrow \Gamma_3$, *and where one of* Γ_1, Γ_2 *contains the sequence* A, B.
- $f = f^{\otimes_i} \circ m$ *for* $f : \Gamma_1, A, B, \Gamma_2 \longrightarrow \Gamma_3$
- $w^{\oplus} = i$
- $g \circ f^{\oplus} = (g \circ f)^{\oplus}$ *for* $g : \Delta_1, C, \Delta_2 \longrightarrow \Delta_3$ *and* $f : \Gamma_1 \longrightarrow \Gamma_2, C, \Gamma_3$, *and where one of* Γ_2, Γ_3 *contains the sequence* A, B.
- $f^{\oplus} \circ g = (f \circ g)^{\oplus}$ *for* $g : \Delta_1 \longrightarrow \Delta_2, C, \Delta_3$ *and* $f : \Gamma_1, C, \Gamma_2 \longrightarrow \Gamma_3$, *and where* Γ_3 *contains the sequence* A, B.
- $f = w \circ f^{\oplus_i}$ *for* $f : \Gamma_1 \longrightarrow \Gamma_2, A, B, \Gamma_3$

We shall leave it as an exercise to show that this is equivalent to the other presentation. However, we must stress that cut elimination does *not* hold for the second presentation of two-tensor-polycategories; the amount of cut built into the rules ($\otimes$ R) and ($\oplus$ L) is necessary to prove cut elimination.

It is straightforward to define the category of polycategories (just keep in mind that we interpret sequents $\Gamma \longrightarrow \Delta$ as maps $\bigotimes \Gamma \longrightarrow \bigoplus \Delta$, and functors should preserve the tensors.) So a functor $F: \mathbf{C} \longrightarrow \mathbf{D}$ is a map $Ob(\mathbf{C}) \longrightarrow Ob(\mathbf{D})$ and a map $M\varphi(\mathbf{C}) \longrightarrow M\varphi(\mathbf{D})$ so that this and the induced map $Ob(\mathbf{C})^* \longrightarrow$

$Ob(\mathbf{D})^*$ commute with *source* and with *target*. A functor between two-tensor-polycategories must preserve the two tensors.

A natural transformation $\alpha: F \longrightarrow G$ assigns a $\mathbf{D}$ morphism $\alpha_A: F(A) \longrightarrow G(A)$ to each singleton sequence A from $\mathbf{C}$, satisfying the usual naturality condition.

We shall denote the 2–category of polycategories by **PolyCat**, and the 2–category of two-tensor-polycategories by $\mathbf{PolyCat}_{\otimes\oplus}$. We then note that the latter is a conservative extension of the former:

Proposition 1.4 *There is a 2-adjunction* $F \dashv U$

$$\begin{tikzcd}\mathbf{PolyCat} \arrow[r, shift left, "F"] & \mathbf{PolyCat}_{\otimes\oplus} \arrow[l, shift left, "U"]\end{tikzcd}$$

whose unit $\mathbf{C} \longrightarrow UF(\mathbf{C})$ *is full for each polycategory* $\mathbf{C}$.

Proof. Given a polycategory $\mathbf{C}$, $F(\mathbf{C})$ is the free two-tensor-polycategory generated by $\mathbf{C}$. That is, close the set $Ob(\mathbf{C})$ under the tensors $\otimes, \oplus$ to obtain the objects of $F(\mathbf{C})$, and take the sequents of $\mathbf{C}$ as non logical axioms, closing under the inference rules to obtain the morphisms of $F(\mathbf{C})$ (actually, you must factor out by the appropriate equivalences first). For a two-tensor-polycategory, U just forgets the two tensor structure.

For a two-tensor-polycategory $\mathbf{D}$, the counit $FU(\mathbf{D}) \longrightarrow \mathbf{D}$ collapses the new tensor structure onto the old. For a polycategory $\mathbf{C}$, the unit $\mathbf{C} \longrightarrow UF(\mathbf{C})$ is the usual inclusion into the free structure. To see that this map is full, we use the cut elimination theorem for two-tensor-polycategories. $F(\mathbf{C})$ has only the sequents of $\mathbf{C}$ as its non logical axioms, so by cut elimination any derivation in $F(\mathbf{C})$ is equivalent to one with cuts restricted to sequents from $\mathbf{C}$. If $\Gamma \longrightarrow \Delta$ is a tensor-free sequent of $F(\mathbf{C})$, (for example, is in the image of the unit,) then any derivation of $\Gamma \longrightarrow \Delta$ is equivalent to a derivation in $\mathbf{C}$, since with the cuts restricted to the tensor-free part of $F(\mathbf{C})$, none of the left or right introduction rules could be used in the derivation (they introduce tensors that could never be eliminated). □

Remark: We believe more: *viz.* that the unit is faithful as well, but as of this writing some details remain to be checked. (For if two derivations of a tensor-free sequent are equivalent in $F(\mathbf{C})$, then this equivalence must be true in $\mathbf{C}$ as well.)

We have not considered the question of units $\top, \bot$ for the tensors $\otimes, \oplus$—these can be added if wanted, together with the obvious rules and equations (as done in [Se89]). Since the point of these units is that they represent "empty" places in the sequents, they are rather redundant in the polycategory context; however, they are useful when we consider weak distributive categories, and we shall feel free to consider $\mathbf{PolyCat}_{\otimes\oplus}$ enriched with these units when this makes matters technically simpler. For an alternate treatment in terms of proof nets, including the units, see [BS91]. Using nets rather than sequents, that paper also manages to solve the coherence problem for weakly distributive categories with units.

2. Weakly distributive categories

2.1. Definition

A **weakly distributive category** $\mathbf{C}$ is a category with two tensors and four weak distribution natural transformations. The two tensors will be denoted by $\otimes$ and $\oplus$ and we shall call $\otimes$ the **tensor** and $\oplus$ the **cotensor**. Each tensor comes equipped with a unit object, an associativity natural isomorphism, and a left and right unit natural isomorphism:

$$
\begin{array}{rcl}
\multicolumn{3}{l}{(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)} \\
a_\otimes & : & (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \\
u^R_\otimes & : & A \otimes \top \longrightarrow A \\
u^L_\otimes & : & \top \otimes A \longrightarrow A \\
\multicolumn{3}{l}{(\oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)} \\
a_\oplus & : & (A \oplus B) \oplus C \longrightarrow A \oplus (B \oplus C) \\
u^R_\oplus & : & A \oplus \bot \longrightarrow A \\
u^L_\oplus & : & \bot \oplus A \longrightarrow A
\end{array}
$$

The four weak distribution transformations shall be denoted by:

$$
\begin{array}{lcl}
\delta^L_L : A \otimes (B \oplus C) & \longrightarrow & (A \otimes B) \oplus C \\
\delta^L_R : A \otimes (B \oplus C) & \longrightarrow & B \oplus (A \otimes C) \\
\delta^R_L : (B \oplus C) \otimes A & \longrightarrow & (B \otimes A) \oplus C \\
\delta^R_R : (B \oplus C) \otimes A & \longrightarrow & B \oplus (C \otimes A).
\end{array}
$$

This data must satisfy certain coherence conditions which we shall discuss shortly. Before doing so we remark that there are three independent symmetries which arise from this data:

[op′] reverse the arrows and swap both the $\otimes$ and $\oplus$ and $\top$ and $\bot$.

$[\otimes']$ reverse the tensor $\otimes$ (so $A \otimes' B = B \otimes A$).

$[\oplus']$ reverse the cotensor $\oplus$.

The notion of a weakly distributive category is preserved by all these symmetries and we shall use this fact to give an economical statement of the required commuting conditions, which are generated by the following diagrams.

Tensors: The two tensor products must each satisfy the usual conditions of a tensor product. (This gives four equations.)

Unit and distribution:

$$\begin{tikzcd}
\top\otimes(A\oplus B) \arrow[d, "\delta^L_L"'] \arrow[dr, "u^L_\otimes"] & \\
(\top\otimes A)\oplus B \arrow[r, "u^L_\otimes\oplus i_B"'] & A\oplus B
\end{tikzcd}$$

(Under the symmetries, this gives eight more equations.)

Associativity and distribution:

$$\begin{tikzcd}
(A\otimes B)\otimes(C\oplus D) \arrow[r, "a_\otimes"] \arrow[dd, "\delta^L_L"'] & A\otimes(B\otimes(C\oplus D)) \arrow[d, "i_A\otimes\delta^L_L"] \\
{} \arrow[r, phantom, "="] & A\otimes((B\otimes C)\oplus D) \arrow[d, "\delta^L_L"] \\
((A\otimes B)\otimes C)\oplus D \arrow[r, "a_\otimes\oplus i_D"'] & (A\otimes(B\otimes C))\oplus D
\end{tikzcd}$$

(Under the symmetries, this gives eight more equations.)

Coassociativity and distribution:

$$\begin{tikzcd}
A\otimes((B\oplus C)\oplus D) \arrow[r, "i_A\otimes a_\oplus"] \arrow[d, "\delta^L_L"'] & A\otimes(B\oplus(C\oplus D)) \arrow[d, "\delta^L_R"] \\
(A\otimes(B\oplus C))\oplus D \arrow[d, "\delta^L_R\oplus i_D"'] \arrow[r, phantom, "="] & B\oplus(A\otimes(C\oplus D)) \arrow[d, "i_B\oplus\delta^L_L"] \\
(B\oplus(A\otimes C))\oplus D \arrow[r, "a_\oplus"'] & B\oplus((A\otimes C)\oplus D)
\end{tikzcd}$$

(Under the symmetries, this gives four more equations.)

Distribution and distribution:

$$\begin{tikzcd}
& (A\oplus B)\otimes(C\oplus D) \arrow[dl, "\delta^L_L"'] \arrow[dr, "\delta^R_R"] & \\
((A\oplus B)\otimes C)\oplus D \arrow[d, "\delta^R_R\oplus i_D"'] & = & A\oplus(B\otimes(C\oplus D)) \arrow[d, "i_A\oplus\delta^L_L"] \\
(A\oplus(B\otimes C))\oplus D \arrow[rr, "a_\oplus"'] & & A\oplus((B\otimes C)\oplus D)
\end{tikzcd}$$

(Under the symmetries, this gives four more equations, for a total of twenty eight.)

2.2. Weakly distributive categories and polycategories

Now we can make the connection between weakly distributive categories and two-tensor-polycategories; essentially these are the same thing. (With Proposition 1.4 this justifies our claiming that weakly distributive categories constitute the essential content of polycategories.) We shall denote the category of weakly distributive categories and functors preserving the tensor and cotensor by **WkDistCat**. (We suppose here that we are using the version of two-tensor-polycategories with units, to correspond to the units in the weakly distributive categories.)

Theorem 2.1 *There is an equivalence of 2-categories*

$$\mathbf{PolyCat}_{\otimes\oplus} \underset{P}{\overset{W}{\rightleftarrows}} \mathbf{WkDistCat}$$

Proof. Given a weakly distributive category $\mathbf{W}$, $P(\mathbf{W})$ is the polycategory with the same set of objects as $\mathbf{W}$, and with morphisms given by: $\Gamma \longrightarrow \Delta$ is a morphism if and only if $\bigotimes \Gamma \longrightarrow \bigoplus \Delta$ is a morphism of $\mathbf{W}$. To check that the cut rule, and the left and right introduction rules, are valid, we use the weak distributivities; for example, we shall illustrate the following instance of cut. Given maps (in $\mathbf{W}$) $C_1 \otimes A \otimes C_2 \xrightarrow{g} C_3$ and $D_1 \xrightarrow{f} D_2 \oplus A \oplus D_3$, we can construct $C_1 \otimes D_1 \otimes C_2 \xrightarrow{g\ {}_1{\circ}_1\ f} D_2 \oplus C_3 \oplus D_3$ as follows (ignoring some instances of associativity for simplicity):

$$\begin{array}{lcl}
C_1 \otimes D_1 \otimes C_2 & \xrightarrow{i\otimes f\otimes i} & C_1 \otimes (D_2 \oplus A \oplus D_3) \otimes C_2 \\
& \xrightarrow{\delta_R^L \otimes i} & (D_2 \oplus (C_1 \otimes (A \oplus D_3))) \otimes C_2 \\
& \xrightarrow{\delta_R^R} & D_2 \oplus ((C_1 \otimes (A \oplus D_3)) \otimes C_2) \\
& \xrightarrow{i\oplus(\delta_L^L \otimes i)} & D_2 \oplus (((C_1 \otimes A) \oplus D_3) \otimes C_2) \\
& \xrightarrow{i\oplus\delta_L^R} & D_2 \oplus ((C_1 \otimes A \otimes C_2) \oplus D_3) \\
& \xrightarrow{i\oplus g\oplus i} & D_2 \oplus C_3 \oplus D_3
\end{array}$$

(Other ways of introducing the distributivities to move the C's next to A are equivalent, by the coherence conditions on the interaction of distributivity with itself and with associativity.)

We must then check that all the equivalences of two-tensor-polycategories follow from the coherence diagrams of weakly distributive categories. This is a frightful but routine exercise. (But note that the extra structure due to the two tensors is easy since ($\otimes$ L) and ($\oplus$ R) are identities. So we really only need check the five equivalences of Definition 1.1.)

Next, given a two-tensor-polycategory **P**, the weakly distributive category $W(\mathbf{P})$ is just the category part of **P**, *viz.* those morphisms whose source and target are singletons. The distributivities are essentially given by the (cut) rule and the axioms ($\otimes$ R) and ($\oplus$ L). For instance, δ^L_R is given as (note the "exchange"):

$$\frac{A,C \longrightarrow A\otimes C \quad B\oplus C \longrightarrow B,C}{A,B\oplus C \longrightarrow B,A\otimes C}$$

And the coherence conditions follow from the equivalences for polycategories.

It is clear from the constructions above that $WP(\mathbf{W})$ is isomorphic to$\mathbf{W}$; indeed they are the same category. And essentially for the same reason, **P** is isomorphic to $PW(\mathbf{P})$. (Essentially, this just depends on the bijection

$$\frac{\Gamma \longrightarrow \Delta}{\bigotimes\Gamma \longrightarrow \bigoplus\Delta}$$

which means that the category part of a two-tensor-polycategory carries all the information of the polycategory.) □

3. Distributive categories

This section has serious errors, corrected in the journal version - available on www.math.mcgill.ca/rags

A weakly distributive category is **symmetric** (resp. $\otimes$–symmetric, $\oplus$–symmetric) in case the tensors are symmetric (resp. the tensor is symmetric with $s_\otimes$, the cotensor with $s_\oplus$) and

$$\begin{tikzcd}
A\otimes(B\oplus C) \arrow[r, "i\otimes s_\oplus"'] \arrow[d, "\delta^L_L"'] & A\otimes(C\oplus B) \arrow[r, "s_\otimes"'] \arrow[d, "\delta^L_R"'] & (C\oplus B)\otimes A \arrow[r, "s_\oplus\otimes i"'] \arrow[d, "\delta^R_R"'] & (B\oplus C)\otimes A \arrow[d, "\delta^R_L"'] \\
(A\otimes B)\oplus C \arrow[r, "s_\oplus"'] & C\oplus(A\otimes B) \arrow[r, "i\oplus s_\otimes"'] & C\oplus(B\otimes A) \arrow[r, "s_\oplus"'] & (B\otimes A)\oplus C
\end{tikzcd}$$

commuting in all squares (resp. those squares which exist).

A weakly distributive category is **cartesian** (resp. $\otimes$–cartesian or $\oplus$–cartesian) if the category is symmetric (resp. $\otimes$–symmetric, $\oplus$–symmetric) with the tensor a product (with $\top$ the final object) and the cotensor a coproduct (with $\bot$ the initial object).

A source of motivation for the study of weak distributivity is the fact that distributive categories are (cartesian) examples. This means that the category of Sets (or any topos) is a cartesian weakly distributive category.

We now verify that distributive categories are a source of examples. An **elementary distributive category** [Co90] has finite products and coproducts such that the comparison map from the coproduct

$$\langle i \times b_0 | i \times b_1 \rangle : A \times B + A \times C \longrightarrow A \times (B + C)$$

is an isomorphism. We shall denote the inverse of $\langle i \times b_0 | i \times b_1 \rangle$ by δ.

Proposition 3.1 *Elementary distributive categories are cartesian weakly distributive categories.*

This proposition is false: an elementary distributive category is cartesian weakly distributive if and only if it is a preorder.

This correction appears in the published journal version of the paper, and on the webpage www.math.mcgill.ca/rags

Proof. Let

δ

then, as + and ed from this. Due to th hat the four basic diag iagrams to check. How the two canonical ways

$$\begin{array}{rcl}
\top \times (A+B) & \longrightarrow & A+B \\
(\bot + A) \times B & \longrightarrow & A \times B \\
(A \times B) \times (C+D) & \longrightarrow & A \times (B \times C) + D \\
(A \times B) \times (C+D) & \longrightarrow & A \times C + B \times D \\
A \times ((C+D)+E) & \longrightarrow & A \times C + (D+E) \\
(A+B) \times (C+D) & \longrightarrow & A + (B \times C) + D
\end{array}$$

For the first of these consider:

$$\begin{tikzcd}
\top \times (A+B) \arrow[r, "p_1"] \arrow[d, "\delta"'] & A+B \\
\top \times A + \top \times B \arrow[r, "i+p_1"'] & \top \times A + B \arrow[u, "p_1+i"']
\end{tikzcd}$$

As $\delta = \langle i \times b_0 | i \times b_1 \rangle^{-1}$ to obtain commutativity it suffices to show:

$$\top \times A \xrightarrow{b_0} \top \times A + \top \times B \xrightarrow{\langle i \times b_0 | i \times b_1 \rangle} \top \times (A+B) \xrightarrow{p_1} A+B = \top \times A \xrightarrow{p_1} A \xrightarrow{b_0} A+B$$

$$\top \times B \xrightarrow{b_1} \top \times A + \top \times B \xrightarrow{\langle i \times b_0 | i \times b_1 \rangle} \top \times (A+B) \xrightarrow{p_1} A+B = \top \times B \xrightarrow{p_1} B \xrightarrow{b_1} A+B$$

which are clear.

For the op′ dual of this we have:

$$\begin{tikzcd}
(\bot + A) \times B \arrow[r, "\delta"] \arrow[d, "b_1^{-1} \times i"'] & \bot \times B + A \times B \arrow[r, "p_0 + i"] & \bot + A \times B \arrow[d, "b_1^{-1}"] \\
A \times B \arrow[rr, equal] & & A \times B
\end{tikzcd}$$

which commutes as $b_1 \times i.\delta.p_0 + i = b_1.p_0 + i = b_1$.

The remaining equations are checked using full distribution applied in two different ways. One can check that the diagram commutes for the components of the coproducts, using the inverses of these distributions, and finally project to obtain the weak distributions. □

It is of some interest to wonder what conditions must be added to a cartesian weakly distributive category to force it to be (elementary) distributive. Demanding that it is cartesian is not sufficient: this can be seen in two ways.

First, an abelian category is a cartesian weakly distributive category as it is a symmetric tensor category on the biproduct. This follows as any braided monoidal category is a weakly distributive category by letting the non permuting weak distributions be the associativity of the tensor and the permuting weak distributions be given by the braiding. Thus, certainly any symmetric monoidal category is a weakly distributive category. Finally, an abelian category is not distributive.

Second, the dual of a distributive category (a codistributive category) is clearly cartesian weakly distributive as the latter is a self-dual notion. However, a codistributive category is not distributive. Indeed, a codistributive category which is simultaneously a distributive category must be a preorder (as the final object is costrict).

In order to obtain a distributive category there must, therefore, be some relationship required between the distribution, projection, and embedding maps. Our first attempt to pin this down is as follows:

Lemma 3.2 *A cartesian weakly distributive category is distributive if and only if the following diagrams*

$$\begin{tikzcd}
A \times B \arrow[r, "i \times b_0"'] \arrow[dr, "b_0"] & A \times (B + C) \arrow[d, "\delta_L^L"] \\
& (A \times B) + C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "i \times b_0"'] \arrow[d, "p_1"'] & A \times (B + C) \arrow[d, "\delta_R^L"] \\
B \arrow[r, "b_0"'] & B + (A \times C)
\end{tikzcd}$$

commute.

Proof. It is easy to check that a distributive category satisfies the two diagrams. For the converse, we must construct the inverse δ of $\langle i \times b_0 | i \times b_1 \rangle$.

We set $\delta = A \times (B + C) \xrightarrow{\Delta \times i} (A \times A) \times (B + C) \xrightarrow{a_\times} A \times (A \times (B + C)) \xrightarrow{i \times \delta_R^L} A \times (B + A \times C) \xrightarrow{\delta_L^L} A \times B + A \times C$. To show that this is the inverse of $\langle i \times b_0 | i \times b_1 \rangle$ we precompose with $i \times b_0$ (by symmetry the same thing will happen on precomposing with $i \times b_1$) and show the result is b_0:

$$\begin{tikzcd}
A \times B \arrow[r, "i \times b_0"] \arrow[d, "\Delta \times i"'] & A \times (B + C) \arrow[d, "\Delta \times i"] \\
(A \times A) \times B \arrow[r, "i \times b_0"] \arrow[d, "a_\times"'] & (A \times A) \times (B + C) \arrow[d, "a_\times"] \\
A \times (A \times B) \arrow[r, "i \times (i \times b_0)"] \arrow[d, "p_1"'] \arrow[dr, "i \times b_0"] & A \times (A \times (B + C)) \arrow[d, "i \times \delta_L^L"] \\
A \times B \arrow[dr, "b_0"] & A \times (A \times B + C) \arrow[d, "\delta_R^L"] \\
 & A \times B + A \times C
\end{tikzcd}$$

where the triangle and parallelogram are the two conditions added. □

An initial object is **strict** in case every map to it is an isomorphism. Notice that, as an abelian category has a zero, it cannot have a strict initial object without being trivial. The initial object of an elementary distributive category, however, is necessarily strict (see [Co90]). This is a difference we now exploit:

Theorem 3.3 *A cartesian weakly distributive category is an elementary distributive category if and only if it has a strict initial object.*

Proof. It suffices to show that the two diagrams above commute in the presence of a strict initial object. To see this consider the two naturality diagrams

$$\begin{tikzcd}
A \times (B + \perp) \arrow[r, "i \times (i + \perp)"'] \arrow[d, "\delta^L_L"'] & A \times (B + C) \arrow[d, "\delta^L_L"] \\
A \times B + \perp \arrow[r, "i + \perp"'] & (A \times B) + C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times (B + \perp) \arrow[r, "i \times (i + \perp)"'] \arrow[d, "\delta^L_R"'] & A \times (B + C) \arrow[d, "\delta^L_R"] \\
B + (A \times \perp) \arrow[r, "i + (i \times \perp)"'] & B + (A \times C)
\end{tikzcd}$$

The first immediately yields the first condition of the lemma. The second due to strictness has the bottom left object isomorphic to B and the horizontal map is then the coproduct embedding. It suffices to prove that the vertical map is essentially a projection. For this consider

$$\begin{tikzcd}
A \times (B + \perp) \arrow[r, "! \times i"] \arrow[d, "\delta^L_R"'] & \top \times (B + \perp) \arrow[d, "\delta^L_R"] \\
B + A \times \perp \arrow[r, "i + (! \times i)"'] & B + \top \times \perp
\end{tikzcd}$$

The lower horizontal map is an isomorphism due to the strictness of the initial object. However, the map across the square is clearly equivalent to a projection. □

4. Adding negation

Definition 4.1 *We define a weakly distributive category with negation to be a weakly distributive category with an object function* $(\_)^{\perp}$, *together with the following parameterized families of maps ("contradiction" and "tertium non datur"):*

$$A^{\perp} \otimes A \xrightarrow{\gamma^L_A} \perp \qquad\qquad A \otimes A^{\perp} \xrightarrow{\gamma^R_A} \perp$$
$$\top \xrightarrow{\tau^L_A} A^{\perp} \oplus A \qquad\qquad \top \xrightarrow{\tau^R_A} A \oplus A^{\perp}$$

which satisfy the following coherence condition

$$\begin{tikzcd}
A\otimes\top \arrow[r, "i\otimes\tau^R"] \arrow[dd, "u^R_\otimes"'] \arrow[ddr, phantom, "="] & A\otimes(A\oplus A^\perp) \arrow[d, "\delta^L_R"] \\
 & A\oplus(A\otimes A^\perp) \arrow[d, "i\oplus\gamma^R"] \\
A & A\oplus\bot \arrow[l, "u^R_\oplus"]
\end{tikzcd}$$

and its eight symmetric forms. Note that the op′ *dual should be modified to switch* A *and* $A^\perp$.

Notice that we have not required that $(\_)^\perp$ be a contravariant functor, but merely that it is defined on objects. Of course, $(\_)^\perp$ is a contravariant functor but this is a consequence of the axioms as we shall see. First we note:

Lemma 4.2 *In a weakly distributive category with negation we have the following adjunctions*

$$A\otimes - \dashv A^\perp\oplus - \qquad A^\perp\otimes - \dashv A\oplus -$$
$$-\otimes B \dashv -\oplus B^\perp \qquad -\otimes B^\perp \dashv -\oplus B$$

corresponding to the following bijections

$$\frac{A\otimes B\longrightarrow C}{B\longrightarrow A^\perp\oplus C} \qquad \frac{A^\perp\otimes B\longrightarrow C}{B\longrightarrow A\oplus C}$$

$$\frac{A\otimes B\longrightarrow C}{A\longrightarrow C\oplus B^\perp} \qquad \frac{A\otimes B^\perp\longrightarrow C}{A\longrightarrow C\oplus B}$$

Proof. We shall treat just the adjunction $-\otimes B\dashv -\oplus B^\perp$ as an illustration. Given a map $A\otimes B\longrightarrow C$, we derive the corresponding map as $A\longrightarrow A\otimes\top\longrightarrow A\otimes(B\oplus B^\perp)\longrightarrow(A\otimes B)\oplus B^\perp\longrightarrow C\oplus B^\perp$. Conversely, given $A\longrightarrow C\oplus B^\perp$, we have $A\otimes B\longrightarrow(C\oplus B^\perp)\otimes B\longrightarrow C\oplus(B^\perp\otimes B)\longrightarrow C\oplus\bot\longrightarrow C$.

In particular, the unit $\eta_A\colon A\longrightarrow(A\otimes B)\oplus B^\perp$ is given by

$$\eta_A\colon A\xrightarrow{u_\otimes^{R-1}}A\otimes\top\xrightarrow{i\otimes\tau^R}A\otimes(B\oplus B^\perp)\xrightarrow{\delta^L_L}(A\otimes B)\oplus B^\perp$$

and the counit $\epsilon_A\colon(A\oplus B^\perp)\otimes B\longrightarrow A$ is given (as the symmetries might suggest) by

$$\epsilon_A\colon(A\oplus B^\perp)\otimes B\xrightarrow{\delta^R_R}A\oplus(B^\perp\otimes B)\xrightarrow{i\oplus\gamma^L}A\oplus\bot\xrightarrow{u^R_\oplus}A$$

We leave checking the triangle identities as an exercise (or see the fuller version of this paper). □

We may now use the adjunctions to define the effect of $(-)^{\perp}$ on maps:

$$\begin{array}{c} A \longrightarrow B \\ \hline \top \otimes A \longrightarrow B \\ \hline \top \longrightarrow B \oplus A^{\perp} \\ \hline B^{\perp} \otimes \top \longrightarrow A^{\perp} \\ \hline B^{\perp} \longrightarrow A^{\perp} \end{array}$$

it is then a matter of verifying that this is functorial, by explicitly giving the "formula" for $B^{\perp} \longrightarrow A^{\perp}$ in terms of $A \longrightarrow B$, and verifying that the appropriate diagrams commute (again this is in the fuller version of this paper).

Furthermore, notice that $(\_)^{\perp}$ is full and faithful as there is a bijection $Hom(A, B) \simeq Hom(B^{\perp}, A^{\perp})$.

Theorem 4.3 *The notions of symmetric weakly distributive categories with negation and $*$-autonomous categories coincide.*

Proof. One direction is more or less automatic now in view of Barr's characterization of $*$-autonomous categories in [Ba79]. That is to say, symmetric weakly distributive categories with negation are $*$-autonomous. Of course, to make the translation to Barr's framework, we must make the following (standard) definition: $A \multimap B = A^{\perp} \oplus B$.

The involutive nature of $(-)^{\perp}$ follows from the lemma straightforwardly: *viz.* the iso $A = A^{\perp\perp}$ is induced by the adjunctions:

$$\begin{array}{c} A \longrightarrow B \\ \hline \top \otimes A \longrightarrow B \\ \hline \top \longrightarrow B \oplus A^{\perp} \\ \hline \top \otimes A^{\perp\perp} \longrightarrow B \\ \hline A^{\perp\perp} \longrightarrow B \end{array}$$

Then we can conclude that $A \multimap B = (A \otimes B^{\perp})^{\perp}$ also.

In either case, it is now easy to verify the essential bijection:

$$\begin{array}{c} A \longrightarrow (B \multimap C^{\perp}) \\ \hline A \longrightarrow B^{\perp} \oplus C^{\perp} \\ \hline A \otimes C \longrightarrow B^{\perp} \\ \hline C \otimes A \longrightarrow B^{\perp} \\ \hline C \longrightarrow B^{\perp} \oplus A^{\perp} \\ \hline C \longrightarrow (B \multimap A^{\perp}) \end{array}$$

Note the use of symmetry here. Of course, if the tensors are not symmetric, then we would have two internal hom's, the other being $B \mathrel{\circ\!\!-} A = B \oplus A^{\perp}$, and the bijection above would end with $C \longrightarrow (A^{\perp} \mathrel{\circ\!\!-} B)$.

Next the other half of the proof: here we give just a brief sketch. It is a straightforward verification to check that $*$-autonomous categories are weakly distributive, though the diagrams can be pretty horrid. We shall just indicate how the weak distribution δ_L^L is obtained, leaving the rest to the faith of the reader.

Defining $A \oplus B = A^\perp \multimap B$, we need $\delta_L^L \colon A \otimes (B^\perp \multimap C) \longrightarrow (A \otimes B)^\perp \multimap C$. While it is possible to give a formula for this morphism, it is perhaps more instructive to give its derivation:

First note that under the functor $(-)^\perp$, the internal hom bijection becomes

$$\frac{C^\perp \longrightarrow (A \otimes B)^\perp}{B \otimes C^\perp \longrightarrow A^\perp}$$

From this it is easy to derive maps $A \otimes (A \otimes B)^\perp \longrightarrow B^\perp \longrightarrow (B^\perp \multimap C) \multimap C$. Then we can use the bijection

$$\frac{A \otimes X \longrightarrow Y \multimap C}{A \otimes Y \longrightarrow X \multimap C}$$

to derive the map $A \otimes (B^\perp \multimap C) \longrightarrow (A \otimes B)^\perp \multimap C$ as needed. □

Remark 4.4 (Planar non commutativity)

The above suggests that (non symmetric) weakly distributive categories with negation provide a natural notion of non symmetric $*$-autonomous categories, and hence of non commutative linear logic (rather, the multiplicative fragment thereof).

However, there is another such notion, building on the planar polycategories of Remark 1.2. In this variant, two different negations are used, ${}^\perp A$ and $A^\perp$. In his presentation at this Symposium, G.L. Mascari described such a system, with the inference rules

$$\frac{\Gamma \vdash \Delta, B}{\Gamma, B^\perp \vdash \Delta} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash A^\perp, \Delta}$$

$$\frac{\Gamma \vdash B, \Delta}{{}^\perp B, \Gamma \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, {}^\perp A}$$

A full account of this syntax appears in [Ab90].

We can modify our presentation to account for this variant. We replace γ_A^L with $\gamma_A^L \colon {}^\perp A \otimes A \longrightarrow \perp$ and τ_A^R with $\tau_A^R \colon \top \longrightarrow A \oplus {}^\perp A$. (And modify the coherence conditions as well, dropping those that no longer make sense.) Then we can derive the adjunctions

$$\begin{array}{ll} - \otimes B^\perp \dashv - \oplus B \qquad & A \otimes - \dashv A^\perp \oplus - \\ {}^\perp B \otimes - \dashv B \oplus - \qquad & - \otimes A \dashv - \oplus {}^\perp A \end{array}$$

corresponding to these rules, and the (natural) isomorphisms $({}^{\perp}A)^{\perp} \simeq A$, ${}^{\perp}(A^{\perp}) \simeq A$. In this context, we would have that $A \multimap B \simeq (A \otimes {}^{\perp}B)^{\perp}, B \circ\!\!- A = B \oplus {}^{\perp}A \simeq {}^{\perp}(B^{\perp} \otimes A)$.

Our original presentation arose in an attempt to describe commutative linear logic: it displays some of the features of the planar non commutative form as well as the commutative form. At this time we feel it is very premature to pronounce definitively on the "best" degree of non commutativity in linear logic, and so we offer only these comments: First, our main observation is that the core of the multiplicative fragment of linear logic may be found in the two tensors, connected by weak distributivity. (We do not believe that the central role played by the weak distributivities—permutative or not—has been sufficiently observed before[2].) Second, to include negation and internal hom, one need only add negation in the most simple minded manner (the internal hom structure follows naturally). Third, the various versions of this fragment may be classified by the degree of the weak distributivity assumed and the nature of the negation added. □

5. Some posetal examples

To conclude, we shall briefly consider some simple examples of weakly distributive categories which are preorders. The beauty of the posetal weakly distributive categories is that one need not check the coherence conditions as all diagrams commute. It suffices to have the weak distributions present. Notice first that, when such a category is cartesian, the initial object is necessarily strict giving:

Lemma 5.1 *All cartesian weakly distributive categories which are preorders are equivalent to distributive lattices.*

Thus, the interesting posetal examples occur when one or both tensors are non cartesian. There are plenty of examples of these. Here are two sources:

- (Droste) Let L be a lattice ordered monoid (that is a set having a commutative, associative, and idempotent operation $x \wedge y$, and an associative operation $x \cdot y$ with unit 1 such that $z \cdot (x \wedge y) = (z \cdot x) \wedge (z \cdot y)$ and $(x \wedge y) \cdot z = (x \cdot z) \wedge (y \cdot z)$) in which every element is less than 1 (so this is the unit of $\wedge$ too) then L is a posetal weakly distributive category. This because:

  $$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z) \leq (x \cdot y) \wedge (1 \cdot z) = (x \cdot y) \wedge z$$

  and similarly for the other weak distributions.

  An example of such an L is the negative numbers. In general one may take the negative portion of any lattice ordered group (free groups can be lattice ordered so that the multiplication need not be commutative).

---

[2] An exception is recent ongoing work of V. dePaiva and J.M.E. Hyland, unpublished but partially presented at Category Theory 1991, Montréal (June 1991), which has among other things pointed out some of the aspects of the distributivities we have in mind here.

- A shift monoid is a commutative monoid $(M, 0, +)$ with a designated invertible element a. This allows one to define a second "shifted" multiplication $x \cdot y = x + y - a$ with unit a for which we have the following identity:

  $$x \cdot (y + z) = (x \cdot y) + z$$

  which clearly is a weak distribution. In this manner a shift monoid becomes a discrete weakly distributive category. Furthermore, it is not hard to show that every discrete weakly distributive category must be a shift monoid.

  This example is also of interest as it suggests that when one inverts the weak distributions (which produces braidings on the tensors), the tensors, which need not be equivalent, are related by a $\oplus$ invertible object. This is, in fact, what happens in general.

It is also of interest to specialize our presentation of $*$-autonomous categories to the case of preorders. Again, only the existence of the maps themselves must be ensured, which gives:

Proposition 5.2 *A preorder is a $*$-autonomous category if and only if it has two symmetric tensors $\otimes$ and $\oplus$ and an object map $(-)^\perp$ such that*

(i) $x \otimes (y \oplus z) \leq (x \otimes y) \oplus z$,

(ii) $x \otimes x^\perp \leq \perp$,

(iii) $\top \leq x \oplus x^\perp$.

Suppose that M is a shift monoid equipped with a map $(-)^\perp$ such that $x + x^\perp = a$ ("tertium non datur") then we have

$$x \cdot x^\perp = x + x^\perp - a = a - a = 0$$

which is "contradiction". So M is a discrete $*$-autonomous category. Note that moreover M is a group, with $-x = x^\perp - a$; in fact shift groups (shift monoids with M a group) are the same as discrete $*$-autonomous categories in this way: $\top = a$, $x^\perp = a - x$, and conversely, a discrete $*$-autonomous category is a group (with respect to $\oplus$, with inverse given by $-x = x^\perp \otimes \perp$), and so a shift group (with $\top$ as designated invertible element). (A curiosity about this example: the initial shift group (also the initial shift monoid) is $\mathcal{Z}$, the integers, under addition with $\top = 1$. This structure also arises when checking the validity of proof nets.)

We can construct similar examples with ordered shift monoids, (for example, $\mathcal{Z}$ as above with the standard order), to get examples of $*$-autonomous posets. Note that a $*$-autonomous ordered shift monoid must be a group, since $x \cdot x^\perp \leq 0$ and $a \leq x + x^\perp$ imply that $x + x^\perp = a$, and so we are in the context above. Note also that by a suitable choice of a we can arrange for the poset to satisfy the mix rule, $x \otimes y \leq x \oplus y$, or its opposite $x \otimes y \geq x \oplus y$, or to be compact $x \otimes y = x \oplus y$.

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