(Marks)

(12) 1. Calculate the following limits (if they exist). Make your answer as informative as possible: if a limit does not exist, say so; if appropriate one-sided limits exist instead, state them explicitly; if any limits are infinite, state this explicitly as well.

(a) $\lim_{x\to+\infty} \dfrac{\sqrt{x^2+4}}{2x+3}$ (b) $\lim_{x\to-2} \sqrt{\dfrac{5x+7}{x^2-16}}$ (c) $\lim_{x\to4} \dfrac{\sqrt{x}-2}{x-4}$

(d) $\lim_{x\to3} \dfrac{2x^2-5x-3}{x^2-x-6}$ (e) $\lim_{x\to-3^+} \dfrac{x^2}{9-x^2}$ (f) $\lim_{x\to1^+} \ln(x-1)$

(4) 2. Refer to the sketch below to evaluate the following. If a value does not exist, state in which way ($+\infty$, $-\infty$, or "does not exist").

(a) $\lim_{x\to-3} f(x) =$

(b) $f(-1) =$

(c) $\lim_{x\to-1^-} f(x) =$

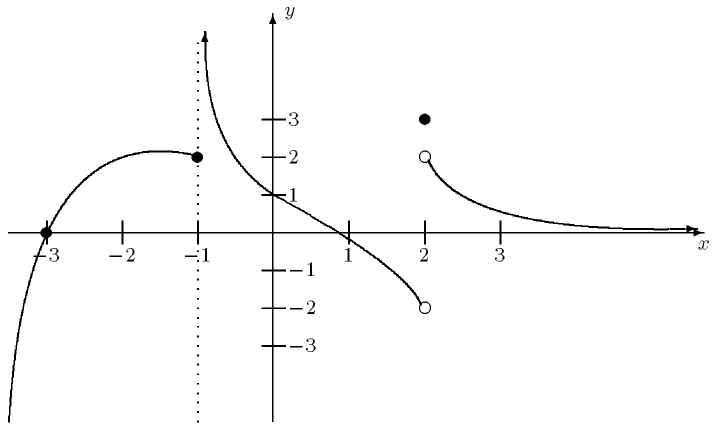
(d) $\lim_{x\to-1^+} f(x) =$

(e) $\lim_{x\to2^-} f(x) =$

(f) $\lim_{x\to2^+} f(x) =$

(g) $f(2) =$

(h) $\lim_{x\to+\infty} f(x) =$



(4) 3. Given the function $f(x) = \begin{cases} \cos(\pi x) & \text{if } x < 1 \\ x^3 - 2 & \text{if } 1 \le x \le 2 \\ 1 - x & \text{if } x > 2 \end{cases}$ use the definition of continuity to determine if f is continuous at (a) $x = 1$ and (b) $x = 2$. If the function is discontinuous at any point, specify whether the discontinuity is removable or not.

(2) 4. Find the value(s) of k for which the function $f(x) = \begin{cases} kx - 1 & \text{if } x < 2 \\ kx^2 & \text{if } x \ge 2 \end{cases}$ is continuous for all x.

(4) 5. State a limit definition of the derivative.

Use this definition to find the derivative of $f(x) = \dfrac{1}{x} - 2$.

(15) 6. For each of the following functions, calculate the derivative $\dfrac{dy}{dx}$. Do **NOT** simplify your answers.

(a) $y = 2x - \dfrac{2}{x} + 2\sqrt{x} - e^{2x} + 2$ (b) $y = \dfrac{4e^x - x^2 + 3}{x + \cos x}$

(c) $y = \tan(2x+1)\,\sin(3x-2)$ (d) $y = \ln^3(5x-2) + \ln 3$

(e) $y = (\sin x)^{2x-3}$

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(3) 7. Given $y = \dfrac{(3x-4)^4\ \sqrt[3]{x^2+2}}{\tan^3 x}$ find $\dfrac{dy}{dx}$ as a function of x using logarithmic differentiation. Do **NOT** simplify your answer.

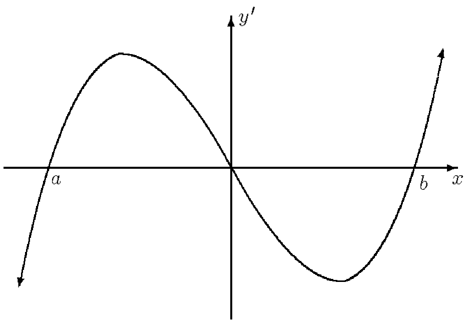
(4) 8. For $f(x) = x\sin(x^2)$, calculate $f''(x)$, and simplify.

(3) 9. Find an equation for the line tangent to the graph of $y = \dfrac{2}{(x-1)^2}$ at the point where $x = 3$.

(3) 10. Given $x + \sqrt{xy} = x^2y - 2$ find $\dfrac{dy}{dx}$ using implicit differentiation.

(3) 11. (a) Give an example of a function whose derivative with respect to x is a non-zero constant k.

(b) For a function $y = f(x)$, the graph of its *derivative* is shown below. Draw a sketch of the graph of the original function f.



(c) Is the following true or false?: $m = 3f(x)$ is the slope of the tangent line to the graph of the function $f(x) = e^{3x}$ for any x.

(4) 12. The position s (in metres) of a particle at time t (seconds) is given by the equation $s = 5t^3 - 30t^2 + 50$. Find the velocity when the acceleration is 0.

(5) 13. A student is filling a conical paper cup at the rate of 3 cm^3/s. If the height of the cup is 10 cm and the diameter of its opening is 6 cm, how fast is the level of the liquid rising when the depth of liquid is 5 cm? (Hint? $V = \frac{1}{3}\pi r^2 h$.)

(4) 14. Determine whether or not the function $S(x) = x + \dfrac{1}{x^2}$ has an absolute (global) minimum value. If it does, what is the minimum value? If you think it does not have an absolute minimum, justify your claim.

(5) 15. For the function $f(x) = \dfrac{x}{x^2+1}$, the first and second derivatives are $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ and $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$. Graph the function $f(x)$, identifying all intercepts, aymptotes, local extrema, and inflection points. Specify intervals where the graph is increasing, decreasing, concave up, and concave down. Show all your work.

(Marks)

(12) 16. Evaluate the following integrals:

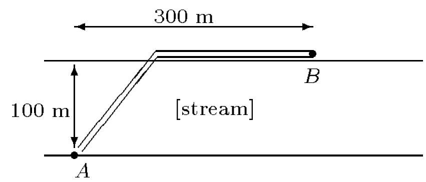
(a) $\displaystyle\int \frac{x^4+2x^3-3x+5}{\sqrt{x}}\,dx$ (b) $\displaystyle\int \left(\frac{1}{t^3}-t^3+e^t-\frac{1}{e^3}\right)\,dt$

(c) $\displaystyle\int_1^2 \left(x^2+\frac{2}{x^2}\right)^2\,dx$ (d) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\theta+\frac{1}{\csc\theta}\right)\,d\theta$

(4) 17. Given that $f'(x) = 3\cos x - x + 2$, and $f(\frac{\pi}{2}) = 3$, find $f(x)$.

(4) 18. The curve $y = 4x - x^3$ intersects the x-axis to form two closed regions: find the total area of these two regions. You may use symmetry, if applicable, to simplify your calculations, but justify your answer with an appropriate explanation.

(5) 19. A pipeline is to be constructed under a stream and along the stream's edge from point A to point B, as illustrated. The cost of construction under the stream is \$500 per meter, and the cost of construction along the stream's edge is \$400 per meter. If cost is to be minimized, how much piping should be constructed under water? (Give your answer correct to the nearest meter.)



(Marks)

The Answers

1. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{7}{5}$ (e) $+\infty$ (f) $-\infty$

2. (a) 0 (b) 2 (c) 2 (d) $+\infty$
   (e) -2 (f) $+2$ (g) 3 (h) 0

3. (a) continuous at $x = 1$ (b) discontinuous at $x = 2$, not removable

4. $k = -1/2$

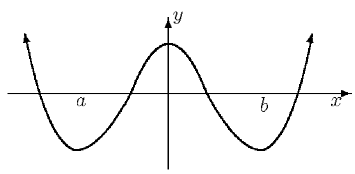
5. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left(\frac{1}{x+h}-2\right)-\left(\frac{1}{x}-2\right)}{h} = \lim_{h\to 0}\frac{x-x-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

6. (a) $y' = 2 + x^{-2} + x^{-1/2} - 2e^{2x}$
   (b) $y' = \dfrac{(4e^x - 2x)(x+\cos x) - (4e^x - x^2 + 3)(1-\sin x)}{(x+\cos x)^2}$
   (c) $y' = 2\sec^2(2x+1)\sin(3x-2) + 3\tan(2x+1)\cos(3x-2)$
   (d) $y' = 3\ln^2(5x-2)\dfrac{5}{5x-2}$
   (e) $y' = (\sin x)^{2x-3}\left(2\ln\sin x + (2x-3)\dfrac{\cos x}{\sin x}\right)$

7. $y' = \dfrac{(3x-4)^4\ \sqrt[3]{x^2+2}}{\tan^3 x}\left(4\dfrac{3}{3x-4} + \dfrac{1}{3}\dfrac{2x}{x^2+2} - 3\dfrac{\sec^2 x}{\tan x}\right)$

8. $f''(x) = 2x(3\cos(x^2) - 2x^2\sin(x^2))$ 9. $y = -\frac{1}{2}x + 2$

10. $y' = \dfrac{4(xy)^{3/2} - y - 2\sqrt{xy}}{x - 2x^2\sqrt{xy}}$

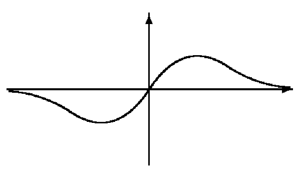
11. (a) Any $y = kx + c$ for k, c constants. (b)
    

    (c) True

12. $v = -60$ m/sec 13. $\frac{4}{3\pi}$ cm/sec

14. The function has no global min (it has a relative min at $x = \sqrt[3]{2}$ only).

15. CP's: $x = \pm 1$: $x = -1$ a min, $x = 1$ a max; PI's: $x = 0, \pm\sqrt{3}$. No VA; HA at $y = 0$. Intercept $(0, 0)$. Graph at right:



16. (a) $\frac{2}{9}x^{9/2} + \frac{4}{7}x^{7/2} - 2x^{3/2} + 10x^{1/2} + C$
    (b) $-\frac{1}{2}t^{-2} - \frac{1}{4}t^4 + e^t - e^{-3}t + C$
    (c) $\frac{341}{30}$ (d) $\frac{7\pi^2}{288} - \frac{1}{2} + \frac{1}{\sqrt{2}}$

17. $f(x) = 3\sin x - \frac{1}{2}x^2 + 2x - \pi + \frac{\pi^2}{8}$ 18. 8 19. 167 m