1. Evaluate the following integrals.

(a) $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

(b) $\displaystyle\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx$

(c) $\displaystyle\int \frac{x+6}{x(x^2+2x+3)}\,dx$

(d) $\displaystyle\int \sin(\ln x)\,dx$

(e) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-4}}\,dx$

(f) $\displaystyle\int \sqrt{\frac{3+x}{3-x}}\,dx$

2. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\sin(2x))}$

(b) $\displaystyle\lim_{x\to \pi/2^-} (\tan x)^{2x-\pi}$

(c) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{3}{e^{3x}-1}\right)$

3. Evaluate each improper integral or show that it diverges.

(a) $\displaystyle\int_0^\infty (-xe^{-x})\,dx$

(b) $\displaystyle\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$

4. Give the solution of the differential equation $\cos x\,\dfrac{dy}{dx} = \sin x\sqrt{y^2+4}$ which satisfies $y=0$ if $x=0$.

5. Find the area of the region bounded by $y_1 = x^3+x^2+3x+1$ and $y_2 = x^3+x+4$.

6. Let $\mathcal{R}$ be the region bounded by $x=0$, $f(x)=1+x$ and $g(x)=x^3+x$. Set up, **but do not evaluate**, an integral which represents the volume obtained by revolving $\mathcal{R}$ about:

(a) the x-axis;

(b) the line $x=3$.

7. Find the arc length function for the curve $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$, taking $\left(\frac{1}{4}, 1\right)$ as the starting point.

8. Determine with justification, whether the sequence $\{a_n\}$ converges or diverges. If a sequence converges, find its limit.

(a) $a_n = \left(\dfrac{3n+1}{3n-1}\right)^n$

(b) $a_n = \dfrac{n^3(2n)!}{(2n+2)!}$

9. For the telescoping series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$,

(a) give a formula for s_n, the sum of the first n terms of the series, and (b) find the sum of the series.

10. Determine whether each series is convergent or divergent. Justify your answers.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n^2+3}}{3n^2+7}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

11. Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers by displaying proper solutions.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{3^{n+1}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{5n+3}}$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{5^n \sqrt{n}}.$$

13. For the function $f(x) = \dfrac{1}{2+x}$, find the Taylor series around $x = 1$. Write the first four terms of the series explicitly, and express the series using appropriate sigma notation.

Solutions

1. a. If $y = \sqrt{\sin x}$ then $2y\,dy = \cos x\,dx$ and $\cos^2 x = 1 - y^4$, so

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = 2\int (1-y^4)\,dy = 2\left(y - \tfrac{1}{5}y^5\right) + a = \tfrac{2}{5}(5 - \sin^2 x)\sqrt{\sin x} + a.$$

b. If $y = \arcsin(x^2)$ then $dy = 2x(1-x^4)^{-1/2}\,dx$, so

$$\int \frac{x \arcsin(x^2)}{\sqrt{1-x^4}}\,dx = \tfrac{1}{2}\int y\,dy = \tfrac{1}{4}\left(\arcsin(x^2)\right)^2 + b.$$

c. The resolution into partial fractions of the integrand is

$$\frac{x+6}{x(x^2+2x+3)} = \frac{2}{x} - \frac{2x+3}{x^2+2x+3},$$

where the coefficient over x is found by inspection (covering and evaluating) and the coefficients over x^2+2x+3 are obtained by comparing the quadratic and constant terms of the numerator. The integral of the second partial fraction is

$$\int\left\{\frac{2x+2}{x^2+2x+3} + \frac{1}{(x+1)^2+2}\right\}dx = \log(x^2+2x+3) + \tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right) + c,$$

and therefore

$$\int \frac{x+6}{x(x^2+2x+3)}\,dx = \log\frac{x^2}{x^2+2x+3} - \tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right) + c.$$

d. Repeated partial integration and absorbtion gives

$$\begin{aligned}\int \sin(\log x)\,dx &= x\sin(\log x) - x\cos(\log x) - \int \sin(\log x)\,dx\\ &= \tfrac{1}{2}x\left(\sin(\log x) - \cos(\log x)\right) + d.\end{aligned}$$

e. If $y = \sqrt{x^2-4}$, then $y^2 = x^2 - 4$, so $y\,dy = x\,dx$ and

$$d\left(\frac{y}{x^2}\right) = \frac{dy}{x^2} - \frac{2y\,dx}{x^3} = \frac{dy}{x^2} - \frac{2(x^2-4)}{x^3 y}\,dx = \frac{8\,dx}{x^3 y} - \frac{dy}{y^2+4}.$$

Therefore,

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \frac{y}{8x^2} + \frac{1}{8}\int\frac{dy}{y^2+4} = \frac{\sqrt{x^2-4}}{8x^2} + \frac{1}{16}\arctan\left(\tfrac{1}{2}\sqrt{x^2-4}\right) + E.$$

f. Multiplying and dividing by $\sqrt{3+x}$ omits -3 from the domain of the integrand, and gives

$$\int \frac{3+x}{\sqrt{9-x^2}}\,dx = 3\arcsin\left(\tfrac{1}{3}x\right) - \int dy = 3\arcsin\left(\tfrac{1}{3}x\right) - \sqrt{9-x^2} + f,$$

where in the second term $y = \sqrt{9-x^2}$, so that $dy = -(x/y)\,dx$.

2. a. Revising the expression in the limit and using the fact that $\lim_{\vartheta\to 0}\frac{\sin\vartheta}{\vartheta} = 1$ gives

$$\lim_{x\to 0+}\frac{\log(\sin x)}{\log(\sin 2x)} = \lim_{x\to 0+}\frac{1 + \log\left(\frac{\sin x}{x}\right)/(\log x)}{1 + \log\left(\frac{\sin 2x}{2x}\right)/(\log 2x)} = 1.$$

b. If $y = \frac{1}{2}\pi - x$ and $z = 1/y$ then

$$\lim_{x\to\frac{1}{2}\pi-}(\tan x)^{2x-\pi} = \lim_{y\to 0+}(\cot y)^{-2y} = \lim_{y\to 0+}\left(\cos y\cdot\frac{y}{\sin y}\right)^{-2y}\cdot\lim_{z\to\infty} e^{2(\log z)/z} = 1$$

by elemenatary properties of the logarithm (the definition of $\log z$ implies that $0 < \log z < z$ if $z > 1$, which immediately gives $0 < (\log z)^a/z^b < (2a/b)^a z^{-b/2}$ for $z > 1$ and $a, b > 0$).

c. Combining terms and using the Maclaurin expansion of the exponential function gives

$$\lim_{x\to 0}\frac{e^{3x} - 1 - 3x}{x(e^{3x}-1)} = \lim_{x\to 0}\frac{\frac{9}{2} + \frac{9}{2}x + \frac{27}{8}x^2 + \cdots}{3 + \frac{9}{2}x + \frac{9}{2}x^2 + \cdots} = \frac{3}{2}.$$

(Alternatively, two applications of l'Hôpital's rule could be used.)

3. a. Partial integration gives

$$\int_0^\infty (-xe^{-x})\,dx = \lim_{t\to\infty}(x+1)e^{-x}\Big|_0^\infty = \lim_{t\to\infty}\left\{\frac{t+1}{e^t} - 1\right\} = -1,$$

via basic properties of the exponential function (the inequality shown in Part c of Question 2 immediately gives $0 < y^a/e^{by^c} < (2a/b)^a e^{-\frac{1}{2}by^c}$ for $a, b, c, y > 0$).

b. Integrating by inspection gives

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \lim_{s\to 1-} 3\sqrt[3]{x-1}\Big|_0^s + \lim_{t\to 1+} 3\sqrt[3]{x-1}\Big|_t^2 = 6.$$

4. For $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, the equation in question is equivalent to

$$\frac{1}{\sqrt{y^2+4}}\frac{dy}{dx} = \tan x, \qquad\text{or}\qquad \log\left(y + \sqrt{y^2+4}\right) = \log(\sec x) + C,$$

which is equivalent to $y + \sqrt{y^2+4} = A\sec x$ (in which $A = e^C$). The initial condition $y = 0$ if $x = 0$ gives $A = 2$, so $y + \sqrt{y^2+4} = 2\sec x$. To express y as a function of x, observe that subtracting y and squaring gives $4\sec^2 x - 4y\sec x = 4$, or $y = \sec x - \cos x = \sin x\tan x$.

5. If $\overline{y} = x^3 + x + 4$ and $\underline{y} = x^3 + x^2 + 3x + 1$ then $\overline{y} - \underline{y} = -x^2 - 2x + 3 = (3+x)(1-x)$, which is positive if $-3 < x < 1$ and vanishes if x is -3 or 1. So the area of the region enclosed by the curves is

$$\int_{-3}^{1}(\overline{y} - \underline{y})\,dx = \int_{-3}^{1}(-x^2 - 2x + 3)\,dx = \left(-\tfrac{1}{3}y^3 - x^2 + 3x\right)\Big|_{-3}^{1} = -\tfrac{28}{3} + 8 + 12 = \tfrac{32}{3}.$$

6. If $\overline{y} = x + 1$ and $\underline{y} = x^3 + x$ then $\overline{y} - \underline{y} = 1 - x^3$, which is positive if $0 < x < 1$ and vanishes if $x = 1$. The solid obtained by revolving $\mathcal{R}$ about the x axis consists of annuli of inner radius $x^3 + x$ and outer radius $x + 1$, for $0 \leqslant x \leqslant 1$, so its volume is equal to

$$\pi\int_0^1\left\{(x+1)^2 - (x^3+x)^2\right\}dx.$$

The solid obtained by revolving $\mathcal{R}$ about the line defined by $x = 3$ consists of concentric cylindrical shells of radius $3 - x$ and radius $1 - x^3$, for $0 \leqslant x \leqslant 1$, so its volume is equal to

$$2\pi\int_0^1 (3-x)(1-x^3)\,dx.$$

7. If $x = \frac{1}{4}y^2 - \frac{1}{2}\log y$, then

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\tfrac{1}{2}y - \tfrac{1}{2}y^{-1}\right)^2 = \tfrac{1}{4}y^2 + \tfrac{1}{2} + \tfrac{1}{4}y^{-2} = \left(\tfrac{1}{2}y + \tfrac{1}{2}y^{-1}\right)^2,$$

and hence

$$\int_1^y \sqrt{1 + \left(\frac{dx}{d\eta}\right)^2}\,dy = \tfrac{1}{2}\int_1^y\left(\eta + \eta^{-1}\right)d\eta = \tfrac{1}{4}(y^2 - 1) + \tfrac{1}{2}\log y,$$

which is the length of the curve between $\left(\frac{1}{4}, 1\right)$ and (x, y) if $y \geqslant 1$, and is -1 times the length of the curve between $\left(\frac{1}{4}, 1\right)$ and (x, y) if $0 < y < 1$.

8. a. Since
$$\lim_{t\to 0}(1+t)^{1/t} = e, \qquad \lim_{n\to\infty}\frac{2n}{3n+1} = \tfrac{2}{3}$$
and
$$a_n = \left(\frac{3n+1}{3n-1}\right)^n = \left(1+\frac{2}{3n-1}\right)^{\frac{3n-1}{2}\cdot\frac{2n}{3n-1}},$$
it follows that $\lim_{n\to\infty} a_n = e^{2/3}$.

b. Since
$$a_n = \frac{n^3(2n)!}{(2n+2)!} = \frac{n^3}{(2n+2)(2n+1)} = \frac{n}{2(1+1/n)(2+1/n)},$$
it follows that the sequence $\{a_n\}$ diverges to ∞.

9. If
$$a_n = \frac{1}{n} - \frac{1}{n+2} \qquad \text{and} \qquad A_n = \frac{1}{n} + \frac{1}{n+1},$$
then $a_n = A_n - A_{n+1}$ for $n \geqslant 1$, and the sum of the first n terms of the series is
$$a_1 + a_2 + \cdots + a_n = A_1 - A_{n+1} = \tfrac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$
Hence, the sum of the series is $\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \frac{3}{2}$.

10. a. If $n \geqslant 1$ then
$$a_n = \frac{\sqrt{n^2+3}}{3n^2+7} > \frac{\sqrt{n^2}}{3n^2+7n^2} = \frac{1}{10n},$$
so the comparison test implies that $\sum a_n$ diverges with the harmonic series.

b. Since $\frac{d}{dx}(x^{-1/4}\log x) = \frac{1}{4}x^{-5/4}(4-\log x)$ is positive if $0 < x < e^4$ and negative if $x > e^4$, it follows that
$$0 < a_n = \frac{\log n}{n^{3/2}} = \frac{\log n}{n^{1/4}} \cdot \frac{1}{n^{5/4}} < \frac{4}{e} \cdot \frac{1}{n^{5/4}},$$
for $n \geqslant 1$. Therefore, the comparison test implies that $\sum a_n$ converges with the p-series $\sum n^{-5/4}$.

11. a. Since $\sum 2^{-n}$ is a convergent geometric series, and
$$a_n = \frac{n!}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)} = 1\cdot\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdots\frac{n}{2n+1} < \left(\frac{1}{2}\right)^n,$$
for $n \geqslant 1$, the comparison test implies that $\sum(-1)^n a_n$ is absolutely convergent.

b. If $n > 3$ then
$$a_n = \frac{n^n}{3^{n+1}} = \frac{1}{3}\cdot\frac{n\cdot n\cdot n\cdot n\cdots n}{3\cdot 3\cdot 3\cdot 3\cdots 3} > \frac{n}{9}, \qquad \text{so} \qquad \lim_{n\to\infty} a_n = \infty.$$
Hence, the vanishing condition implies that $\sum(-1)^n a_n$ is divergent.

c. If $n \geqslant 1$ then
$$a_n = \frac{1}{\sqrt{5n+3}} \geqslant \frac{1}{\sqrt{5n+3n}} = \frac{\sqrt{2}}{4}\cdot\frac{1}{\sqrt{n}} > 0,$$
so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$. On the other hand,
$$a_n = \frac{1}{\sqrt{5n+3}} > \frac{1}{\sqrt{5n+8}} = a_{n+1}$$
if $n \geqslant 1$, and $\lim a_n = 0$, so the alternating series test implies that $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum \cos(n\pi)a_n = \sum(-1)^n a_n$ is conditionally convergent.

12. If $x \neq -1$ and
$$\alpha_n = \frac{(-1)^n(x+1)^n}{5^n\sqrt{n}}, \qquad \text{then} \qquad \lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim\frac{1}{5\sqrt{1+1/n}}|x+1| = \tfrac{1}{5}|x+1|,$$
so the ratio test implies that $\sum \alpha_n$ is absolutely convergent if $|x+1| < 5$, *i.e.*, $-6 < x < 4$, and divergent if $x < -6$ or $x > 4$. If $x = -6$ then $\sum \alpha_n = \sum n^{-1/2}$ is a divergent p-series ($p = \frac{1}{2} \leqslant 1$), and if $x = 4$ then $\sum \alpha_n = \sum(-1)^n n^{-1/2}$, which is convergent by the alternating series test ($n^{-1/2} > (n+1)^{-1/2}$ if $n \geqslant 1$, and $\lim n^{-1/2} = 0$). Therefore, the raduis of convergence of $\sum \alpha_n$ is 5, and the interval of convergence of $\sum \alpha_n$ is $(-6, 4]$.

13. From the expansion $1/(1+t) = \sum_{k=0}^{\infty}(-1)^k t^k$ (a geometric series), it follows that
$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{3}\cdot\frac{1}{1+\frac{1}{3}(x-1)} = \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}(x-1)^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{3^{k+1}}(x-1)^k \\ &= \tfrac{1}{3} - \tfrac{1}{9}(x-1) + \tfrac{1}{27}(x-1)^2 - \tfrac{1}{81}(x-1)^3 + \cdots,\end{aligned}$$
which is valid if $\frac{1}{3}|x-1| < 1$, or equivalently, $-2 < x < 4$.