

1. (6 points) Given the graph of f below, evaluate the following expressions. If appropriate use ∞ , $-\infty$, or “does not exist” where appropriate.

(a) $\lim_{x \rightarrow 5^-} f(x)$

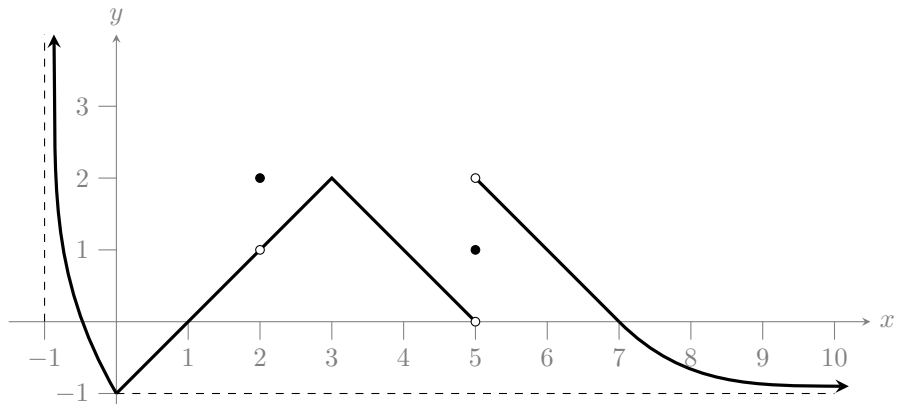
(b) $\lim_{x \rightarrow 5^+} f(x)$

(c) $f(f(5))$

(d) $\lim_{x \rightarrow -1^+} f(x)$

(e) $\lim_{h \rightarrow 0} \frac{f(5/4 + h) - f(5/4)}{h}$

(f) $f''(4)$



2. (10 points) Evaluate the following limits.

(a) $\lim_{x \rightarrow 1} \frac{3x^2 - x - 2}{x^2 - x}$

(b) $\lim_{x \rightarrow \infty} \left(\frac{1+x}{6+5x^2} \right) \left(\frac{5+7x^3}{2-5x^2} \right)$

(c) $\lim_{\theta \rightarrow 0} \frac{\tan(6\theta)}{\sin(8\theta)}$

(d) $\lim_{x \rightarrow \infty} (\sqrt{4x^2 + 3x - 2} - 3x)$

(e) $\lim_{x \rightarrow 1^+} \frac{x+1}{x - |2-3x|}$

3. (4 points) Let

$$f(x) = \begin{cases} \frac{1}{k+1-x} & \text{if } x \leq 3, \\ \sqrt{\frac{x^2 - 5x + 6}{k(x-3)}} & \text{if } x > 3. \end{cases}$$

Find all values of k that make the function $f(x)$ continuous at $x = 3$.

4. (3 points) Find an equation of the normal line to the curve $y = \frac{x^2}{x-2}$ at the point with x -coordinate equal to 3.

5. (4 points) Find the derivative of $f(x) = \frac{1}{2x+1}$ using the limit definition of the derivative.

6. (15 points) Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \frac{\sqrt[3]{x}}{2} + \frac{2}{x+1} - 3^x + \cos(e^2)$

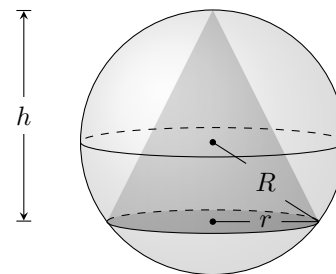
(b) $y = \frac{x}{x+1} + \ln\left(\frac{2}{x}\right)$

(c) $y = \csc^2(3x^2) + \ln(4-x) + xe^{3x^2}$

(d) $y = (x-1)(2+x)^{2x}$

(e) $\sin(x-y) = xy$

7. (2 points) A particle moves along a straight line with its position at time t given by $s(t) = t^{2/3}(20-t)$. What is the distance travelled by the particle during the time interval $[1, 27]$?
8. (4 points) Let $f(x) = e^x(x^3 - 3x^2 + 6x + 2)$.
- (a) Justify that $f(x)$ has a root in the interval $(-1, 0)$.
- (b) Justify that $f(x)$ has only one root in the interval $(-1, 0)$.
9. (5 points) Find all the points on the graph of the equation $x^4 + y^4 + 2 = 4xy^3$ at which the tangent line is horizontal.
10. (5 points) A plane, flying in a straight line at a constant altitude of 4 km, passes directly over a telescope tracking it. At a certain moment the angle between the telescope's line of sight and the ground is $\pi/3$ and is decreasing at a rate of $1/2$ radians per minute. How fast is the plane travelling at that moment?
11. (4 points) Find the absolute extrema of $f(x) = 15 + 12x - x^3$ on $[1, 4]$.
12. (5 points) Find the height of the right circular cone of largest volume that can be inscribed in a sphere of radius R . ($V = \frac{1}{3}\pi r^2 h$)



13. (10 points) Given $f(x) = \frac{6 - 2e^x}{e^x + 1}$, $f'(x) = -\frac{8e^x}{(e^x + 1)^2}$, $f''(x) = \frac{8e^x(e^x - 1)}{(e^x + 1)^3}$,

find (if any):

- (a) domain of f ,
- (b) x and y intercept(s),
- (c) equations of all asymptotes,
- (d) intervals on which f is increasing or decreasing,
- (e) local (relative) extrema,
- (f) intervals of upward or downward concavity,
- (g) inflection point(s).
- (h) On the next page, sketch the graph of f . Label all intercepts, asymptotes, extrema, and points of inflection.
14. (2 points) Given that $f'(x) = 3\sin(x) + \frac{1}{\pi}$ and $f(3\pi/4) = 0$, find $f(x)$.

15. (5 points) Compute the definite integral $\int_1^4 (x^2 - x + 1) dx$ as a limit of Riemann sums.

Note that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

16. (2 points) Find a number b and a function f such that

$$\int_2^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{3 + \frac{4i}{n}} \left(\frac{4}{n}\right).$$

17. (3 points) Evaluate $\int_{-2}^2 (|x| + \sqrt{4 - x^2}) dx$ by interpreting it in terms of areas.

18. (9 points) Evaluate the following integrals.

(a) $\int \frac{(\sqrt{x} - 1)^2}{x} dx$

(b) $\int \left(e^{x+1} + \frac{\sec x \tan x}{2} + 3 \sec^2 x \right) dx$

(c) $\int_1^e \left(\frac{2}{t} + \frac{1}{e} \right) dt$

19. (2 points) Given $F(x) = \int_x^{2x} \left(\frac{\sin t}{t} \right) dt$ find $F'(x)$.

Exam Solutions

1. (a) 0 (b) 2 (c) 0 (d) ∞ (e) 1 (f) 0

2. (a) $\lim_{x \rightarrow 1} \frac{3x + 2}{x} = 5$

(b) $\lim_{x \rightarrow \infty} \frac{7x^4}{-25x^4} = -\frac{7}{25}$

(c) $\lim_{\theta \rightarrow 0} \frac{\sin(6\theta)}{6\theta} \cdot \frac{8\theta}{\sin(8\theta)} \cdot \frac{3}{4 \cos(6\theta)} = \frac{3}{4}$

(d) $\lim_{x \rightarrow \infty} (2x - 3x) = -\infty$

(e) $\lim_{x \rightarrow 1^+} \frac{x + 1}{x + (2 - 3x)} = \lim_{x \rightarrow 1^+} \frac{x + 1}{-2(x - 1)} = -\infty$

3. $\frac{1}{k-2} = \sqrt{\frac{1}{k}} \Rightarrow k = 1$ (extraneous), $k = 4$

4. $\frac{dy}{dx} = \frac{x^2 - 4x}{(x-2)^2} \Rightarrow -\frac{dx}{dy} \Big|_{x=3} = \frac{(3-2)^2}{4(3) - (3)^2} = \frac{1}{3}$

$y(3) = 9$

$y = \frac{1}{3}(x - 3) + 9 \Rightarrow y = \frac{1}{3}x + 8$

5. $\lim_{h \rightarrow 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h} = \frac{-2}{(2x+1)^2}$

6. (a) $\frac{dy}{dx} = \frac{1}{6\sqrt[3]{x^2}} - \frac{2}{(x+1)^2} - \ln(3)3^x$

(b) $\frac{dy}{dx} = \frac{1}{(x+1)^2} - \frac{1}{x}$

(c) $\frac{dy}{dx} = -12x \csc^2(3x^2) \cot(3x^2) - \frac{1}{4-x} + e^{3x^2} + 6x^2 e^{3x^2}$

(d) $\frac{dy}{dx} = \left(\frac{1}{x-1} + 2 \ln(2+x) + \frac{2x}{2+x}\right) (x-1)(2+x)^{2x}$

(e) $\frac{dy}{dx} = \frac{\cos(x-y) - y}{x + \cos(x-y)}$

7. $s'(t) = \frac{40-5t}{3\sqrt[3]{t}} \Rightarrow$ critical points: $t = 0, 8$

$s'(t) > 0$ for $1 < t < 8$ and $s'(t) < 0$ for $8 < t < 27 \Rightarrow$ distance = $(s(8) - s(1)) + (s(8) - s(27)) = 128$ units.

8. (a) Note $f(x)$ is continuous on \mathbb{R} , and $f(-1) = \frac{-8}{e} < 0$, $f(0) = 2 > 0$. Conclude by IVT.

(b) Suppose $f(x)$ has a second root in $(0, -1)$. $f(x)$ is differentiable on \mathbb{R} , so Rolle's Theorem would assure that $f'(x) = 0$ at some point in $(0, -1)$ between these roots. But $f'(x) = e^x(x^3 + 8) > 0$ on $(0, -1)$. So there can be no second root on $(0, -1)$.

9. $4x^3 + 4y^3y' = 4y^3 + 12xy^2y' \Rightarrow y' = \frac{4(x^3 - y^3)}{4y^2(3x - y)}$

$y' = 0 \Rightarrow x = y \Rightarrow 2y^4 + 2 = 4y^2 \Rightarrow y = \pm 1$ Points: $(1, 1), (-1, -1)$

10. $\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{4}{x}\right) \Rightarrow \sec^2(\theta) \frac{d\theta}{dt} = -\frac{4}{x^2} \frac{dx}{dt}$

$\frac{d\theta}{dt} = -\frac{1}{2}, \theta = \frac{\pi}{3}$

$\tan(\pi/3) = \frac{4}{x} \Rightarrow x = \frac{4}{\sqrt{3}}$

$\frac{dx}{dt} = 4 \cdot \frac{1}{2} \cdot \frac{16/3}{4} = \frac{8}{3}$ km/min

11. $f'(x) = 12 - 3x^2 \Rightarrow$ critical points: $x = \pm 2$.

$f(1) = 26, f(2) = 31, f(4) = -1 \Rightarrow$ Local max: $(2, 31)$, Local min: $(4, -1)$.

12. We have $r^2 = R^2 - (h - R)^2 = 2Rh - h^2$, so

$V(h) = \frac{\pi}{3}(2Rh^2 - h^3)$

$V'(h) = \frac{\pi}{3}(4Rh - h^3)$

Critical points: $h = R, h = \frac{4}{3}R$

Check: $V''(\frac{4}{3}R) = \frac{\pi}{3}(4R - 6(\frac{4}{3}R)) < 0$ so $h = \frac{4}{3}R$ yields the maximal volume.

13. Domain: \mathbb{R}

x -int: $(\ln(3), 0)$,

y -int: $(0, 2)$

V.A.: None

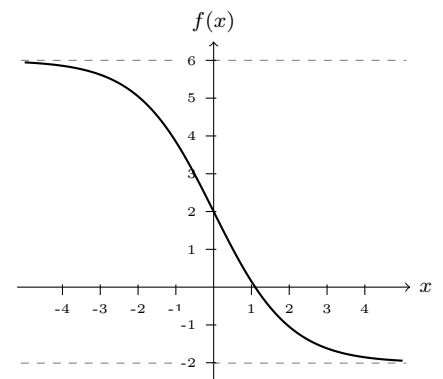
H.A.: $y = 6$ on the left, $y = -2$ on the right.

Decrease: \mathbb{R} , Critical points: None.

Possible inflection points: $x = 0$.

Concave up: $(0, \infty)$, Concave down: $(-\infty, 0)$,

Inflection point at $(0, 2)$.



14. $f(x) = -3 \cos(x) + \frac{x}{\pi} + C$

$-3 \cos(3\pi/4) + \frac{3\pi}{4\pi} + C = 0 \Rightarrow C = -\frac{3+6\sqrt{2}}{4}$

$f(x) = -3 \cos(x) + \frac{x}{\pi} - \frac{3+6\sqrt{2}}{4}$

15.

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(1 + \frac{3}{n}i\right)^2 - \left(1 + \frac{3}{n}i\right) + 1 \right) \frac{3}{n} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{3}{n}i + \frac{9}{n^2}i^2\right) \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{3n}{n} + \frac{9n(n+1)}{2n^2} + \frac{27n(n+1)(2n+1)}{6n^3} \right) \\ &= 3 + \frac{9}{2} + 9 = \frac{33}{2}\end{aligned}$$

16. $b = 6$, $f(x) = \sqrt{1+x}$

17. $\int_{-2}^2 |x| dx + \int_{-2}^2 \sqrt{4-x^2} dx = 4 + 2\pi$

18. (a) $\int \frac{x - 2x^{1/2} + 1}{x} dx = \int 1 - 2x^{-1/2} + \frac{1}{x} dx = x - 4\sqrt{x} + \ln|x| + C$

(b) $e^{x+1} + \frac{\sec(x)}{2} + 3 \tan(x) + C$

(c) $(2 \ln|t| + \frac{t}{e}) \Big|_1^e = (2 \ln(e) + \frac{e}{e}) - (2 \ln|1| + \frac{1}{e}) = 3 - \frac{1}{e}$

19. $\frac{\sin(2x) - \sin(x)}{x}$