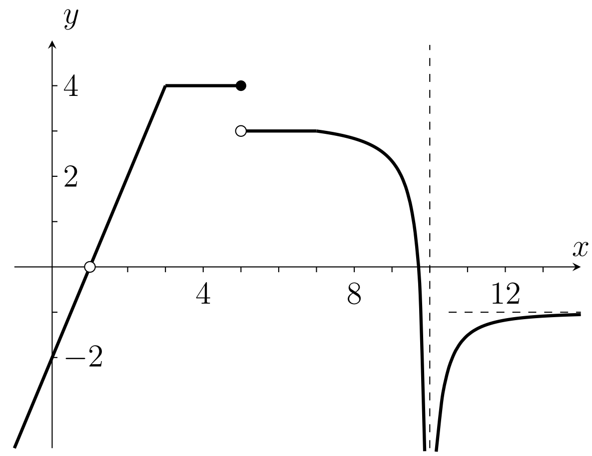
1. (6 points) Given the graph of f below, evaluate each of the following. Use ∞, $-\infty$ or "does not exist" where appropriate.



(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 10} f(x)$

(c) $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

(d) $\lim_{x\to \infty} f\left(\frac{1}{x}\right)$

(e) $\lim_{x\to \infty} f(x)$

(f) $f'(5)$

2. (12 points) Evaluate each of the following limits.

(a) $\lim_{x\to 2} \dfrac{x^3-7x^2+10x}{2x^2-3x-2}$

(b) $\lim_{x\to 5} \dfrac{\frac{1}{x-8}+\frac{1}{3}}{x^2-25}$

(c) $\lim_{x\to 0} \dfrac{\sin(5x)}{x^2+10x}$

(d) $\lim_{x\to \frac{3\pi}{2}} \dfrac{\cos x}{1-\sqrt{1-\cos x}}$

(e) $\lim_{x\to \infty} \sqrt[3]{\dfrac{7+x^2-8x^3}{x^3-x+\pi}}$

(f) $\lim_{x\to 2^-} \dfrac{|x-2|}{(x-2)^2}$

3. (5 points) Let $f(x) = \begin{cases} ax-b & \text{if } x \leqslant -1, \\ 2x^2+3ax+b & \text{if } -1 < x \leq 1 \\ 4 & \text{if } x > 1. \end{cases}$

Find all values of a and b so that $f(x)$ is continuous for all values of x.

4. (4 points) Use the limit definition of the derivative to find $f'(x)$, where $f(x) = x + \dfrac{1}{x}$.

5. (15 points) Find $\dfrac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y = \dfrac{x^2 - \sqrt[3]{x^4} + \pi\sqrt{x}}{\sqrt{x}}$

(b) $y = e^{5x^2} - 6x\,4^x - 3\ln(7x+1) - \log_2(\cos x)$

(c) $y = \left(\dfrac{x^2+2}{x^2-2}\right)^{10}$

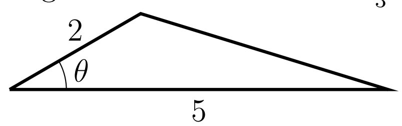
(d) $y = \dfrac{\sqrt{x^2+2}\,\sqrt[3]{x^3+3}}{\sqrt[4]{x^4+4}}$ *Use logarithmic differentiation.*

(e) $e^{xy} + 7 = y \tan x$

6. (3 points) How many tangent lines to the curve $f(x) = \dfrac{x}{2x-1}$ pass through the point $(-7, 1)$? At which points do these tangent lines touch the curve?

7. (4 points) Prove that the equation $e^x = -x + 2$ has *exactly* one real root.

8. (5 points) Two sides of a triangle are 2 cm and 5 cm long respectively, and the angle between them is increasing at a rate of $\frac{1}{2}$ rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides is $\frac{\pi}{3}$.



9. (4 points) Find the absolute extrema of $f(x) = \dfrac{2x}{e^x}$ on $[0, 10]$.

10. (10 points) Given

$$f(x) = (x+3)^{1/3}(x-1)^{2/3}, \quad f'(x) = \frac{3x+5}{3(x+3)^{2/3}(x-1)^{1/3}} \quad \text{and} \quad f''(x) = \frac{-32}{9(x+3)^{5/3}(x-1)^{4/3}},$$

and that $\sqrt[3]{3} \approx 1.4$ and $\dfrac{4^{4/3}}{3} \approx 2.1$, find:

(a) The domain of $f(x)$.

(b) All x and y intercepts.

(c) All vertical and horizontal asymptotes.

(d) All intervals on which $f(x)$ is increasing or decreasing.

(e) All local (relative) extrema.

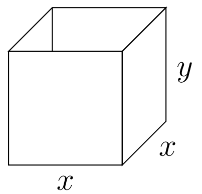
(f) All intervals of upward and downward concavity.

(g) All inflection points.

On the next page, sketch the graph of $f(x)$. Label all intercepts, asymptotes, extrema, and points of inflection.

11. (3 points) Suppose that $f(x)$ is a differentiable function such that $f(x) > 0$ and $f'(x) < 0$ for all real values of x.
Show that $g(x) = \dfrac{1-f(x)}{1+f(x)}$ is an increasing function.

12. (5 points) A square piece of sheet metal is to be made into an open-topped box by cutting squares from its corners and folding up the sides. If the box must have a volume of $2\,\mathrm{m}^3$, what should the dimensions of the box be to minimize the area of the original square sheet?



13. (4 points) Find $f(x)$ if $f''(x) = -\sin x$, $f(0) = -1$ and $f(\frac{\pi}{2}) = \pi$.

14. (12 points) Evaluate each of the following integrals.

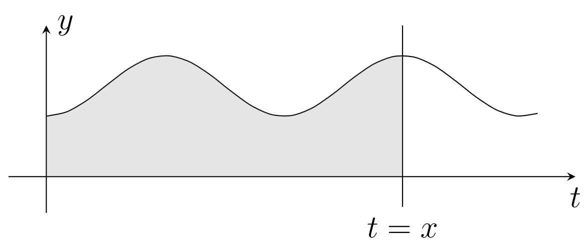
(a) $\displaystyle\int \left(x^5 + 5^x + \ln 5\right)\, dx$

(b) $\displaystyle\int_1^e \left(1 - \frac{1}{x}\right)^2 dx$

(c) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan x\,(\cos x - \sec x)\, dx$

(d) $\displaystyle\int \left(\sqrt{2x} + 2x\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

15. (3 points) Let $g(x)$ be the area of the region enclosed by the curves $y = 1 + \sin^2 t$, the t-axis, $t = 0$, and the line $t = x$, as shown below. Find $g'(x)$.



16. (a) (1 point) Express the integral $\displaystyle\int_0^1 \left(4x - x^2\right) dx$ as a limit of Riemann sums, taking sample points to be right endpoints.

(b) (4 points) Use summation formulæ and basic properties of limits to evaluate the integral from part (a).
Note that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
No marks if you use the Fundamental Theorem of Calculus to evaluate the integral.

Answers

1.(a) 0 (b) $-\infty$ (c) 2 (d) -2 (e) -1 (f) DNE 2.(a) $-\frac{6}{5}$ (b) $-\frac{1}{90}$ (c) $\frac{1}{2}$ (d) 2 (e) -2 (f) ∞

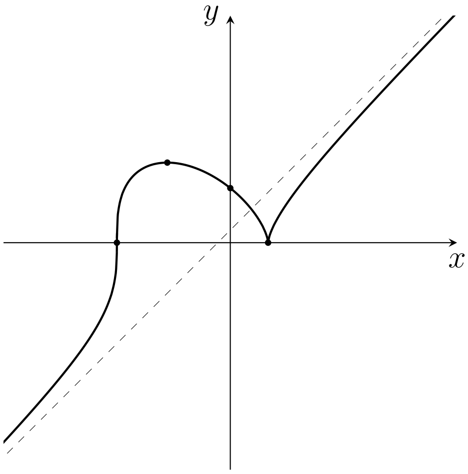
3. $a = \frac{3}{4}$, $b = -\frac{1}{4}$ 4. $1 - \frac{1}{x^2}$ 5.(a) $\frac{3}{2}x^{1/2} - \frac{5}{6}x^{-1/6}$ (b) $10xe^{5x^2} - 6(4^x) - 6x\,4^x \ln 4 - \frac{21}{7x+1} + \frac{\tan x}{\ln 2}$

(c) $\frac{-80x(x^2+2)^9}{(x^2-2)^{11}}$ (d) $\frac{\sqrt{x^2+2}\,\sqrt[3]{x^3+3}}{\sqrt[4]{x^4+4}}\left(\frac{x}{x^2+2} + \frac{x^2}{x^3+3} - \frac{x^3}{x^4+4}\right)$ (e) $\frac{y\sec^2 x - ye^{xy}}{xe^{xy} - \tan x}$ 6. $\left(-1, \frac{1}{3}\right)$ and $\left(3, \frac{3}{5}\right)$

7. Use IVT + Rolle's Thm 8. $\frac{5}{4}\,\mathrm{cm}^2/\mathrm{s}$ 9. Abs Min: $(0,0)$, Abs Max: $\left(1, \frac{2}{e}\right)$ 10.(a) $\mathbb{R}$

(b) x-int: $x = -3$ and 1, y-int: $y = \sqrt[3]{3}$ (c) None (d) Inc: $\left(-\infty, -\frac{5}{3}\right)$ and $(1,\infty)$, Dec: $\left(-\frac{5}{3}, 1\right)$

(e) LMax: $(-\frac{5}{3}, \frac{4^{4/3}}{3})$, LMin: $(1,0)$ (f) CU: $(-\infty,-3)$, CD: $(-3,1)$ and $(1,\infty)$ (g) $(-3,0)$



11. $g'(x) = -\frac{2f'(x)}{(1+f(x))^2} > 0$ 12. $x = 2\,\mathrm{m}$, $y = \frac{1}{2}\,\mathrm{m}$ 13. $f(x) = \sin x + 2x - 1$

14.(a) $\frac{x^6}{6} + \frac{5^x}{\ln 5} + x\ln 5 + C$ (b) $e - 2 - \frac{1}{e}$ (c) $\frac{7\sqrt{3} - 9\sqrt{2}}{6}$ (d) $\frac{2\sqrt{2}}{3}x^{3/2} + \frac{4}{5}x^{5/2} + 2x^{1/2} + C$

15. $g'(x) = 1 + \sin^2 x$ 16.(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(4\frac{i}{n} - \left(\frac{i}{n}\right)^2\right)\frac{1}{n}$ (b) $\frac{5}{3}$