(Marks)

(4) 1. Refer to the sketch below to evaluate the following. If a value does not exist, state in which way ($+\infty$, $-\infty$, or "does not exist").

(a) $\lim_{x\to -3} f(x) =$

(b) $f(-1) =$

(c) $\lim_{x\to -1^-} f(x) =$

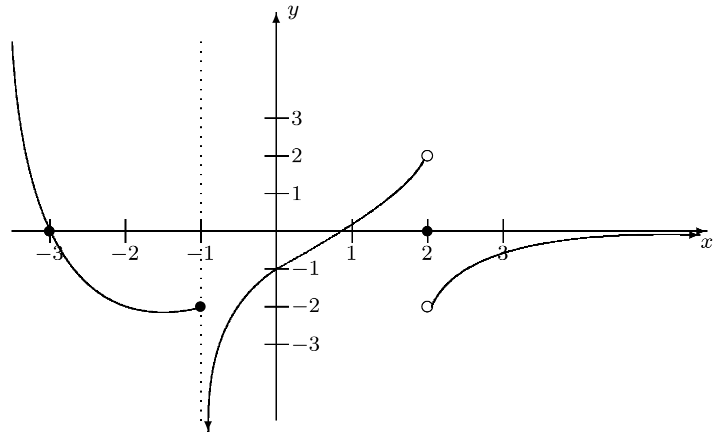
(d) $\lim_{x\to -1^+} f(x) =$

(e) $\lim_{x\to 2^-} f(x) =$

(f) $\lim_{x\to 2^+} f(x) =$

(g) $f(2) =$

(h) $\lim_{x\to +\infty} f(x) =$



(9) 2. Evaluate the limits. Use the symbols $-\infty$ or $+\infty$ where appropriate.

(a) $\lim_{x\to +\infty} \dfrac{\sqrt{x^2+4}}{2-3x}$ (b) $\lim_{x\to -2} \dfrac{x^2+5x+6}{x^2-x-6}$ (c) $\lim_{x\to 4^-} \dfrac{|x-4|}{x-4}$

(d) $\lim_{x\to 3} \dfrac{2-\sqrt{7-x}}{3-x}$ (e) $\lim_{x\to 0} \dfrac{\sin^2 x}{3x}$

(4) 3. Given the function $f(x) = \begin{cases} x-1 & \text{if } x<0 \\ x^3-2 & \text{if } 0\le x\le 1 \\ \cos(\pi x) & \text{if } x>1 \end{cases}$

use the definition of continuity to determine if f is continuous at (a) $x=0$ and (b) $x=1$. For each discontinuity, specify what type of discontinuity it is.

(2) 4. Find the value(s) of k for which the function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}+k & \text{if } x<1 \\ x^2 & \text{if } x\ge 1 \end{cases}$ is continuous for all x.

(4) 5. State a limit definition of the derivative.

Use this definition to find the derivative of $f(x) = \dfrac{1}{2-x}$.

(15) 6. For each of the following functions, calculate the derivative $\dfrac{dy}{dx}$. You do **NOT** have to simplify your answers.

(a) $y = 2x^4 - \dfrac{2}{\sqrt[3]{x}} + \dfrac{\sqrt{x}}{3} - \ln 2$ (b) $y = \ln^3(ax+b)$ where a, b are constants

(c) $y = \tan^3(x+1)\cos^5(3x-2)$ (d) $y = \dfrac{e^{x^2}-x^2+3}{x+\sec(3x)}$

(e) $y = (\sin 2x)^{2x^4-3}$

(3) 7. Determine all x values where the graph of $y = x^3 e^{3x}$ has a horizontal tangent line.

(4) 8. The position s (in metres) of a particle at time t (seconds) is given by the equation $s = \sqrt{t} + \frac{1}{\sqrt{t}}$. Find the velocity when the acceleration is 0 m/sec^2.

(Marks)

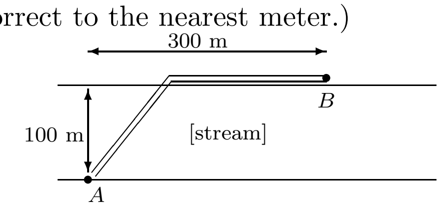
(4) 9. Find an equation for the line tangent to the graph of $y = \dfrac{4}{(x-2)^2}$ at the point where $x = 4$.

(4) 10. Given $\sqrt{x^4 + y^4} = x^2 y^2$ find $\dfrac{dy}{dx}$ using implicit differentiation.

(5) 11. A student is filling a conical paper cup at the rate of 3 cm^3/s. If the height of the cup is 10 cm and the diameter of its opening is 6 cm, how fast is the level of the liquid rising when the depth of liquid is 5 cm? (Hint: $V = \frac{1}{3}\pi r^2 h$.)

(4) 12. Does the function $S(x) = 2x + \dfrac{1}{x^2}$ have an absolute (global) minimum; does it have an absolute (global) maximum? In each case, find it if it exists, and if it does not exist, justify your claim.

(5) 13. A pipeline is to be constructed under a stream and along the stream's edge from point A to point B, as illustrated. The cost of construction under the stream is \$500 per meter, and the cost of construction along the stream's edge is \$400 per meter. If cost is to be minimized, how much piping should be constructed under water? (Give your answer correct to the nearest meter.)



(8) 14. For the function $f(x) = \dfrac{x^2}{x^2+3}$, sketch the graph of $y = f(x)$, identifying all intercepts, aymptotes, local extrema, and inflection points. Specify intervals where the graph is increasing, decreasing, concave up, and concave down. Show all your work.

(12) 15. Evaluate the following integrals:

(a) $\displaystyle\int \frac{x^3 - 4x^2 + 3x - 1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \left(\frac{1}{t^4} - t^4 + e^t - \frac{1}{e^4}\right) dt$

(c) $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^2 dx$

(d) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sin\theta + \frac{1}{\sin^2\theta}\right) d\theta$

(3) 16. Using the Fundamental Theorem of Calculus, find the derivative of the function

$$f(x) = \int_0^{x^2} \frac{dt}{1 + t + \sin(t)}$$

(3) 17. Given that $f'(x) = x + \frac{1}{x} - 2$, and $f(1) = 3$, find $f(x)$.

(3) 18. Suppose $f(x)$ is a function with the properties

$$\int_0^1 f(x)\,dx = 6 \qquad \int_0^2 f(x)\,dx = 4 \qquad \int_2^5 f(x)\,dx = 1$$

find the values of:

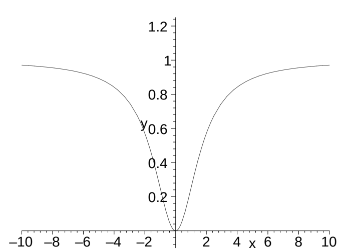
(a) $\displaystyle\int_1^2 f(x)\,dx =$ (b) $\displaystyle\int_5^1 f(x)\,dx =$ (c) $\displaystyle\int_3^3 f(x)\,dx =$

(4) 19. Find the area between the curves $y = x + \sec^2 x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{4}$.

(Marks)

Answers

1. (a) 0 (b) -2 (c) -2 (d) $-\infty$ (e) 2 (f) -2 (g) 0 (h) 0
2. (a) $-\frac{1}{3}$ (b) $-\frac{1}{5}$ (c) -1 (d) $-\frac{1}{4}$ (e) 0
3. (a) a gap discontinuity (b) continuous
4. $k=-1$
5. $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{1}{2-(x+h)}-\frac{1}{2-x}\right)=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{(2-x)-(2-x-h)}{(2-x-h)(2-x)}\right)=\lim\limits_{h\to 0}\frac{1}{h}\frac{h}{(2-x-h)(2-x)}=\frac{1}{(2-x)^2}$
6. (a) $y'=8x^3+\frac{2}{3}x^{-4/3}+\frac{1}{6}x^{-1/2}$
   (b) $y'=3\ln^2(ax+b)\frac{a}{ax+b}$
   (c) $y'=3\tan^2(x+1)\sec^2(x+1)\cos^5(3x-2)-15\tan^3(x+1)\cos^4(3x-2)\sin(3x-2)$
   (d) $y'=\frac{(2x\,\mathrm{e}^{x^2}-2x)(x+\sec 3x)-(\mathrm{e}^{x^2}-x^2+3)(1+3\sec 3x\tan 3x)}{(x+\sec 3x)^2}$
   (e) $y'=(\sin 2x)^{2x^4-3}\left(8x^3\ln(\sin 2x)+(2x^4-3)\frac{2\cos 2x}{\sin 2x}\right)$
7. $x=-1,0$
8. $v=\frac{1}{3\sqrt{3}}$ (at $t=3$)
9. $y=-x+5$
10. $$\frac{dy}{dx}=\frac{xy^2-\frac{x^3}{\sqrt{x^4+y^4}}}{\frac{y^3}{\sqrt{x^4+y^4}}-x^2y}=\frac{x^3(y^4-1)}{y^3(1-x^4)}$$
11. $\frac{4}{3\pi}$ cm/s
12. No min, no max (because $S\to-\infty$ as $x\to-\infty$, and $S\to+\infty$ as $x\to 0$, and also as $x\to+\infty$ — so in each case the extremum is impossible).
13. 167 meters
14. Intercept $(0,0)$, no VA, HA at $y=1$, local min at $(0,0)$, PIs at $(-1,\frac{1}{4})$, $(1,\frac{1}{4})$. Increasing for $x>0$, decreasing for $x<0$; CU for $-1<x<1$, CD for $x<-1$ and $x>1$. The graph:



15. (a) $\frac{2}{7}x^{7/2}-\frac{8}{5}x^{5/2}+2x^{3/2}-2x^{1/2}+C$
    (b) $-\frac{1}{3}t^{-3}-\frac{1}{5}t^5+\mathrm{e}^t-\frac{1}{\mathrm{e}^4}t+C$
    (c) $\frac{29}{6}$ (d) $\frac{1}{2}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}$
16. $f'(x)=\frac{2x}{1+x^2+\sin(x^2)}$
17. $f(x)=\frac{1}{2}x^2+\ln|x|-2x+\frac{9}{2}$
18. (a) -2 (b) 1 (c) 0
19. $\frac{\pi^2}{32}+1$