(Marks)

1. Consider the curve passing through the origin at $t = 0$, given by parametric equations $x = x(t)$ and $y = y(t)$, $0 \le t \le 3$, satisfying

$$\frac{dx}{dt} = 1 - t^2 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

(2) (a) Find the parametric equations of the curve.

(4) (b) Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$. Use these to sketch the graph of the curve for $0 \le t \le 3$. Carefully indicate horizontal and vertical tangents, and intervals where the curve is concave up and where it is concave down. Label relevant points with their t values as well as their xy coordinates.

(4) (c) Find the length of the curve. (Hint: the algebra does work out well if you simplify carefully!)

(4) 2. Use an appropriate power series to approximate $\sqrt{9.1}$ to within $\pm 10^{-4}$.

(4) 3. For the function $f(x) = \dfrac{e^x - e^{-x}}{2}$ find the Maclaurin series for $f'(x)$.

(5) 4. Express $\displaystyle\int_0^x \frac{t^2}{1-t^4}\,dt$ as a power series; what is its interval of convergence?

(6) 5. Identify and sketch the following surfaces. Show all your work, including appropriate traces and intercepts.

(a) $z^2 - x^2 - y^2 = 1$

(b) $\rho = \csc \varphi$

(10) 6. A curve is defined by $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + \ln(t)\,\boldsymbol{j} + t^2\,\boldsymbol{k}$

(a) Find the unit tangent and unit normal vectors $\boldsymbol{T}, \boldsymbol{N}$, and the curvature κ.

(b) Find the tangential and normal components of acceleration a_T, a_N.

(Hint: the algebra does work out well if you simplify carefully!)

(6) 7. Is the following function continuous at the origin? Be sure to justify your answer.

$$f(x) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

(6) 8. For the function $f(x, y) = \ln\left((x-1)^2 + y^2\right)$,

(a) find the directional derivative of f at $(2, -2)$ in the direction of the origin $(0, 0)$;

(b) find the direction of maximum increase in f at the point $(2, -2)$ as well as the magnitude of that increase there;

(c) find a unit vector $\boldsymbol{u}$ so that the directional derivative $f_{\boldsymbol{u}}(2, -2) = 0$.

(6) 9. Suppose a function $z = f(x, y)$ is defined by the equation $\cos(xy) - xz^2 - yz^3 = 3$, and $f(0, 2) = -1$, so that $P_0\,(0, 2, -1)$ is on the surface given by this function.

(a) Find the tangent plane to the surface at P_0.

(b) Calculate $\boldsymbol{\nabla} f(0, 2)$; use this (or part (a) if you prefer) to find an approximation to $f(0.1, 1.9)$. (Hint: calculate df for the appropriate dx, dy.)

(Marks)

(6) 10. Suppose $f(t)$ is a differentiable function; show that $z = yf(x^2 - y^2)$ is a solution of the equation $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = \frac{xz}{y}$.

(6) 11. Find and classify the critical points of $f(x,y) = x^3 + 3xy^2 + 3y^2 - 15x + 2$.

(7) 12. Use Lagrange Multipliers to find maximum and minimum values of $f(x,y,z) = 3x - y - 3z$ subject to the constraints $x + y - z = 0$ and $x^2 + 2z^2 = 18$.

(3) 13. Suppose $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are three linearly independant vectors in R^3. The vector $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ lies in which of the following three planes?: the plane spanned by $\boldsymbol{a}, \boldsymbol{b}$; the plane spanned by $\boldsymbol{a}, \boldsymbol{c}$; the plane spanned by $\boldsymbol{b}, \boldsymbol{c}$; or none of these. Briefly justify your claim.

(3) 14. Without calculating any antiderivatives, what is the value of $\int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} dy\, dx$?

(Hint: sketch the region.)

15. Evaluate the following.

(6) (a) $\int_0^2 \int_{y/2}^1 y\, e^{x^3}\, dx\, dy$

(6) (b) $\iiint_{\mathcal{E}} (1+z)\, dV$, where $\mathcal{E}$ is the solid region inside the cone $z = 2\sqrt{x^2+y^2}$, above the xy plane, and below the plane $z = 6$.

(6) 16. Sketch the solid region $\mathcal{S}$ between the the cone $z = \sqrt{x^2+y^2}$ and the xy plane, inside the cylinder $x^2 + y^2 = 1$.
Set up the triple integral necessary to find the volume of $\mathcal{S}$ using cylindrical coordinates;
set up this triple integral using spherical coordinates as well.
Evaluate *one* of these integrals to determine the volume of $\mathcal{S}$.